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On the weakly nonlinear solitary pressure waves in a blood-filled thin elastic tube

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ABSTRACT

The propagation of weakly solitary pressure waves in a fluid-filled elastic tube have been investigated. The reductive perturbation method has been employed to derive the modified Korteweg-de Vries equation for small but finite amplitude. The effect of the final inner radius of the tube r_f on the basic properties of the soliton wave were discussed.

KEYWORDS

Nonlinear pressure waves;
Fluid;
Thin elastic tube.

INTRODUCTION

The theoretical modeling and experimental investigation in the Biosciences are the elucidation of the underlying biological processes that result in a particular observed phenomenon[1]. In blood vessels experiments, it is found that the flow velocity depends on the elastic properties of the vessel wall and they propagate towards the periphery with a characteristic diagram[2]. The propagation of pressure waves in fluid-filled distensible tubes has been theoretically studied by several researchers[3-5]. Yomosa[6] investigated the nonlinear propagation of localized solitary waves in large blood vessels. He found that the wave pulses of pressure and flow propagating through the arteries can be described as soliton waves excited by cardiac ejections of blood and the features of the pulse wave such as "peaking" and "steepening" are interpreted in the viewpoint of soliton. Later, Shoucri and Shoucri studied the application of the method of characteristics of shock waves in blood flow in the Aorta[7]. By using various asymptotic methods[8], Demiray[9] studied the motion of weakly nonlinear pressure waves in a thin nonlinear elastic tube filled with an incompressible fluid. He proved that, when viscosity of blood is neglected, the dynamics are governed by the Korteweg-de Vries equation. Theoretical investigations for the blood waves by the weakly nonlinear theory have been developed[10-13]. It is shown that the dynamics of the blood waves are governed by the KdV or modified KdV equations. The solitary wave model gives a plausible explanation for the peaking and steeping of pulsatile waves in arteries[6]. Recently, many authors have been drived KdV or modified KdV equations to investigate the propagation of solitary waves in plasma physics[14-17]. The major topic of this work is to study the propagation of pressure waves in weakly nonlinear waves in a fluid-filled elastic tube by means of modified KortewegdeVries equation (mKdV). This paper is organized as follows: in section 2, we present the basic set of fluid equations governing our model. In section 3, long wave approximation is used to drive mKdV equation and solution for mKdV equation are obtained. In Section 4, some discussions and conclusions are given.

BASIC EQUATIONS

To drive the equation of motion of the tube, let us consider a circular cylindrical long tube of radius R_0 with a uniform inner pressure P_0, the axial stretch ratio λ_z and r_0 be the radius of the cylindrical tube after such a finite static deformation. The position vector of a generic point of the tube may be described by.

$$\mathbf{r} = (\mathbf{r}_0 + \mathbf{u}^*)\mathbf{e}_r + \mathbf{z}^*\mathbf{e}_z, \quad \mathbf{z}^* = \boldsymbol{\lambda}_z \mathbf{Z}, \tag{1}$$

Where $u^*(z^*,t^*)$, e_r, e_θ, e_z, z^*, are a finite time time dependent radial displacement, unite base vectors in the cylindrical polar coordinates, the spatial coordinate in the intermediate configuration and the axial coordinate of a point in the undeformed configuration respectively. The axial displacement in neglected in view of the external tethering, the unit tangnt vector *t* to the meridional curve and the unit exrerior normal vector *n* to the deformed membrane are given by

$$\mathbf{t} = \frac{1}{\Lambda}\left(\frac{\partial \mathbf{u}^*}{\partial \mathbf{z}^*}\mathbf{e}_r + \mathbf{e}_z\right), \quad \mathbf{n} = \frac{1}{\Lambda}\left(\mathbf{e}_r - \frac{\partial \mathbf{u}^*}{\partial \mathbf{z}^*}\mathbf{e}_z\right), \tag{2}$$

Where Λ is defined by.

$$\Lambda = \left[1 + \left(\frac{\partial \mathbf{u}^*}{\partial \mathbf{z}^*}\right)^2\right]^{1/2}, \tag{3}$$

The stretches in the axial and circumferential directions may be given as follows:

$$\lambda_1 = \lambda_z \Lambda, \quad \lambda_2 = \lambda_\theta + \mathbf{u}^*/\mathbf{R}_0, \tag{4}$$

Where $\lambda_\theta = r_0/R_0$ is the stretch ratio in the circumferential direction after finite static deformation.

Let F_1 and F_2 be the membrane forces acting along each unit length of the meridional and circumferential curves of the tube respectively. The equation of motion of the tube in the radial direction is given by

$$\frac{\partial}{\partial z^*}\left\{\frac{\mu}{\left[1+(\partial u^*/\partial z^*)^2\right]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\frac{\partial u^*}{\partial z^*}\right\} - \frac{\mu}{\lambda_z R_0}\frac{\partial \Sigma}{\partial \lambda_2} + \left(\lambda_\theta + \frac{u^*}{R_0}\right)\frac{P^*}{H} = \frac{\rho_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}, \tag{5}$$

Where μ is the shear modulus of the tube material, $\mu\Sigma$ is the strain energy density function, H is the initial tube thickness, p^* is the fluid pressure and p_0 is the mass density of the tube material. In order to complete the field equations one must know the value of the fluid pressure p^*. Therefore, Equation (5) is to be complemented with the equations governing the blood fluid. Blood is known to be an incompressible non-Newtonian fluid. The main factor for blood to behave like a non-Newtonian fluid is the deformability of red blood cells and the level of cell concentration (hematocrit ratio). When blood flow in arteries the red cells move to the central region of the artery and, thus, the hematocrit ratio is reduced near the arterial wall, where the shear rate is quite high, as can be seen from Poiseuille flow. In another word experimental observations indicate that when the shear rate is high, blood behaves like a Newtonian fluid. The ratio of the viscous terms to the nonlinear term is $\frac{\mu_v}{\rho_f}\frac{\partial^2 \upsilon^*}{\partial z^2} / \upsilon^* \frac{\partial \upsilon^*}{\partial z} \approx \frac{\mu_v}{\rho_f V' T' \upsilon^*} \approx O(10^{-5})$, considering $\mu_v = 0.04$p and $\rho_v = 1.05$g / cm^3. Therefore, the viscous effect in comparison to the nonlinear effect can be neglected. Based on these observations, we assume that blood is an incompressible inviscid fluid whose equations of axially symmetrical motion in the cylindrical polar coordinates are given by

$$\frac{\partial V_r^*}{\partial r} + \frac{V_r^*}{r} + \frac{\partial V_{z^*}^*}{\partial z^*} = 0, \tag{6}$$

$$\frac{\partial V_r^*}{\partial t^*} + V_r^*\frac{\partial V_r^*}{\partial r} + V_{z^*}^*\frac{\partial V_r^*}{\partial z^*} + \frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial r} = 0, \tag{7}$$

$$\frac{\partial V_{z^*}^*}{\partial t^*} + V_r^*\frac{\partial V_{z^*}^*}{\partial r} + V_{z^*}^*\frac{\partial V_{z^*}^*}{\partial z^*} + \frac{1}{\rho_f}\frac{\partial \bar{P}}{\partial z^*} = 0, \tag{8}$$

Where V_r^*; $V_{z^*}^*$ are the fluid velocity components in the radial and axial directions, respectively, P_f is the mass density of the fluid and $\bar{P}$ is the fluid pressure function. These field equations must satisfy the following boundary conditions:

$$V_r^*\Big|_{r=r_f} = \frac{\partial u^*}{\partial t^*} + \frac{\partial u^*}{\partial z^*}V_z^*\Big|_{r=r_f}, \qquad \bar{P}\Big|_{r=r_f} = P^*, \tag{9}$$

Here $\bar{P}$ is the fluid pressure function, P_f is the fluid mass density and r_f is the final inner radius of the tube. Where $\lambda_\theta = (r_0/R_0)$ is the stretch ratio in the circumferential direction after the finite static deformation. Where, from Equation (5) it is obtained that.

$$P^* = \frac{\rho_0 R_0 H}{\lambda_z (R_0\lambda_\theta + u^*)}\frac{\partial^2 u^*}{\partial t^{*2}} + \frac{\mu H}{\lambda_z (R_0\lambda_\theta + u^*)}\frac{\partial \Sigma}{\partial \lambda_2} - \frac{R_0 H}{(R_0\lambda_\theta + u^*)}\frac{\partial}{\partial z^*}\left\{\frac{\mu}{\left[1+\left(\frac{\partial u^*}{\partial z^*}\right)^2\right]^{1/2}}\frac{\partial \Sigma}{\partial \lambda_1}\frac{\partial u^*}{\partial z^*}\right\}, \tag{10}$$

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In general, the strain energy density Σ is a function of λ_1 and λ_2. For our purposes, we shall assume that Σ is analytic in λ_1 and λ_2 and can be expanded into power series of the following form:

$$P^* = \frac{\beta_1 \mu H}{R_0^2} u^* + \frac{\rho_0 H}{\lambda_\theta \lambda_z} \frac{\partial^2 u^*}{\partial t^{*2}} - \alpha_0 \mu H \frac{\partial^2 u^*}{\partial z^{*2}} + \frac{\beta_2 \mu H}{R_0^3} u^{*2} - \frac{\rho_0 H}{\lambda_\theta^2 \lambda_z R_0} u^* \frac{\partial^2 u^*}{\partial t^{*2}} - \frac{\alpha_1 \mu H}{R_0}\left(\frac{\partial u^*}{\partial z^*}\right)^2 + \left(\frac{\alpha_0 \mu H}{\lambda_\theta R_0} - \frac{2\alpha_1 \mu H}{R_0}\right) u^* \frac{\partial^2 u^*}{\partial z^{*2}} + \frac{\mu H}{R_0} p_0, \tag{11}$$

Where the coefficients $\alpha_0, \dots, \beta_2$ are defined by.

$$\alpha_0 = \frac{1}{\lambda_\theta \lambda_z} \frac{\partial \Sigma}{\partial \Lambda}\bigg|_{u^*=0}, \quad \beta_1 = \frac{R_0}{\lambda_\theta \lambda_z}\left(R_0 \frac{\partial^2 \Sigma}{\partial u^{*2}} - \frac{1}{\lambda_\theta} \frac{\partial \Sigma}{\partial u^*}\right)\bigg|_{u^*=0},$$
$$\beta_2 = \frac{R_0^3}{2\lambda_\theta \lambda_z} \frac{\partial^3 \Sigma}{\partial u^{*3}}\bigg|_{u^*=0} - \frac{\beta_1}{\lambda_\theta}, \tag{12}$$

Equations (6)–(11) give sufficient relations to determine the unknowns u^*, V_r^*, V_z^* and $\overline{P}$.

LONGWAVE APPROXIMATION

The reductive perturbation method is used to study the propagation of small-but-finite amplitude. Let us introduce the following types of stretched coordinates[18]:

$$\xi = \varepsilon\left(z^* - g t^*\right), \qquad \tau = \varepsilon z^*, \tag{13}$$

Where ε is a small parameter measuring the smallness of nonlinearity, dissipation and dispersion, g is the phase velocity in the longwave approximation to be determined later. By introducing transformation (13) into Eqns. (6)-(11) there are founded that. All physical quantities appearing in equations (6-10) are expanded as power series in ε about their equilibrium values as:

$$V_r^* = \sum_{n=1}^{\infty} \varepsilon^{1+n} V_r^{*(n)}(\xi,\tau,r), \qquad V_z^* = \sum_{n=1}^{\infty} \varepsilon^{n} V_z^{*(n)}(\xi,\tau,r),$$
$$\overline{P} = \sum_{n=1}^{\infty} \varepsilon^{n} \overline{P_{(n)}}(\xi,\tau,r), \qquad u^* = \sum_{n=1}^{\infty} \varepsilon^{n} u_n^*(\xi,\tau,r), \tag{14}$$

We impose the boundary conditions that as

$$\overline{P}\Big|_{r=r_f} = \varepsilon^2\left(\frac{\rho_0 H g^2}{\lambda_\theta \lambda_z} - \alpha_0 \mu H\right)\frac{\partial^2 u^*}{\partial \xi^2} + \frac{\beta_1 \mu H}{R_0^2} u^* - \varepsilon^6 \alpha_0 \mu H \frac{\partial^2 u^*}{\partial \tau^2} - 2\varepsilon^4 \alpha_0 \mu H \frac{\partial^2 u^*}{\partial \xi \partial \tau}$$
$$+ \frac{\beta_2 \mu H}{R_0^3} u^{*2} - \varepsilon^2 \frac{\rho_0 H g^2}{\lambda_\theta^2 \lambda_z R_0} u^* \frac{\partial^2 u^*}{\partial \xi^{*2}} - \frac{\alpha_1 \mu H}{R_0}\left[\varepsilon^2\left(\frac{\partial u^*}{\partial \xi}\right)^2 + 2\varepsilon^4 \frac{\partial u^*}{\partial \xi}\frac{\partial u^*}{\partial \tau} + \varepsilon^6\left(\frac{\partial u^*}{\partial \tau}\right)^2\right]$$
$$+ \left(\frac{\alpha_0 \mu H}{\lambda_\theta R_0} - \frac{2\alpha_1 \mu H}{R_0}\right) u^* \left(\varepsilon^2 \frac{\partial^2 u^*}{\partial \xi^2} + \varepsilon^6 \frac{\partial^2 u^*}{\partial \tau^2} + \varepsilon^4 \frac{\partial^2 u^*}{\partial \xi \partial \tau}\right) + \frac{\mu H}{R_0} p_0,$$
$$V_r^*\Big|_{r=r_f} = -\varepsilon g \frac{\partial u^*}{\partial \xi} + \varepsilon\left(\frac{\partial u^*}{\partial \xi} + \varepsilon^2 \frac{\partial u^*}{\partial \tau}\right) V_z^*\Big|_{r=r_f}, \tag{15}$$

Substituting (13) and (14) into equations (6-8) and (11) and equating coefficients of like powers of ε. Then, from the lowest-order equations in ε, the following results are obtained:

$$\frac{1}{\rho_f}\frac{\partial \overline{P}_1}{\partial r} = 0,$$
$$-g\frac{\partial V_z^{*(1)}}{\partial \xi} + \frac{1}{\rho_f}\frac{\partial \overline{P}_1}{\partial \xi} = 0,$$
$$\frac{\partial V_r^{*(1)}}{\partial r} + \frac{V_r^{*(1)}}{r} + \frac{\partial V_z^{*(1)}}{\partial \xi} = 0, \tag{16}$$

And the boundary conditions.

$$\overline{P}_1\Big|_{r=r_f} = \frac{\beta_1 \mu H}{R_0^2} u_1^*, \qquad V_r^{*(1)}\Big|_{r=r_f} = -g\frac{\partial u_1^*}{\partial \xi}, \tag{17}$$

From the solution of the sets (16) and (17) there are obtained that.

$$\overline{p}_1 = P(\xi,\tau), \qquad u_1^* = \frac{R_0^2}{\beta_1 \mu H} P(\xi,\tau),$$
$$V_z^{*1} = \frac{1}{g\rho_f} P(\xi,\tau), \quad V_r^{*1} = -\frac{1}{2g\rho_f}\frac{\partial P(\xi,\tau)}{\partial \xi} r,$$
$$g^2 = \frac{\beta_1 \mu H r_f}{2\rho_f R_0^2} \tag{18}$$

Where $P(\xi,\tau)$ is an unknown function whose governing equation will be obtained later and g is the phase velocity in the longwave approximation. Considering now the coefficients of $O(\varepsilon^2)$, we derive with the aid of (18) the following set of equations:

$$\frac{1}{\rho_f}\frac{\partial \overline{P}_2}{\partial r} = 0,$$
$$-g\frac{\partial V_z^{*(2)}}{\partial \xi} + \frac{1}{g^2 \rho_f^2} P \frac{\partial P}{\partial \xi} + \frac{1}{\rho_f}\frac{\partial \overline{P}_2}{\partial \xi} = 0,$$
$$\frac{\partial V_r^{*(2)}}{\partial r} + \frac{V_r^{*(2)}}{r} + \frac{\partial V_z^{*(2)}}{\partial \xi} = 0, \tag{19}$$

And the boundary conditions can be written as.

$$\overline{P}_2\Big|_{r=r_f} = \frac{\beta_1 \mu H}{R_0^2} u_2^* + \frac{\beta_2 R_0}{\beta_1^2 \mu H} P^2,$$
$$V_r^{*(2)}\Big|_{r=r_f} = -g\frac{\partial u_2^*}{\partial \xi} + \frac{R_0^2}{g\rho_f \beta_1 \mu H} P \frac{\partial P}{\partial \xi}, \tag{20}$$

From Eqn. (19-a) and Eqn. (20-a) we have

$$\frac{\beta_1 \mu H}{R_0^2}\frac{\partial u_2^*}{\partial r} + \frac{\beta_2 R_0}{\beta_1^2 \mu H}\frac{\partial}{\partial r}(P^2) = 0,$$

Integrate this equation w.r.t.r

$$u_2^* = -\frac{\beta_2 R_0^3}{\beta_1^3 \mu^2 H^2} P^2 \quad (21)$$

From Eqn. (19-b) we have.

$$-g\frac{\partial V_z^{*(2)}}{\partial \xi} + \frac{1}{g^2 \rho_f^2} P\frac{\partial P}{\partial \xi} - \frac{\beta_2 R_0}{\beta_1^2 \mu H}\frac{\partial}{\partial \xi}(P^2) + \frac{\beta_2 R_0}{\beta_1^2 \mu H}\frac{\partial}{\partial \xi}(P^2) = 0$$

Integrate this equation w.r.t.ξ

$$V_z^{*(2)} = \frac{1}{2g^3 \rho_f^2}(P^2) \quad (22)$$

Considering now the coefficients of O(ε^3), we derive with the aid of (18), (21) and (22) the following set of equations:

$$\frac{r}{2\rho_f}\frac{\partial^2 P}{\partial \xi^2} + \frac{1}{\rho_f}\frac{\partial \overline{P_3}}{\partial r} = 0 ,$$
$$-g\frac{\partial V_z^{*(3)}}{\partial \xi} + \frac{3P^2}{2g^4\rho_f^3}\frac{\partial P}{\partial \xi} + \frac{1}{\rho_f}\frac{\partial \overline{P_3}}{\partial \xi} + \frac{1}{\rho_f}\frac{\partial P}{\partial \tau} = 0, \quad (23)$$
$$\frac{\partial V_r^{*(3)}}{\partial r} + \frac{V_r^{*(3)}}{r} + \frac{\partial V_z^{*(3)}}{\partial \xi} + \frac{1}{g\rho_f}\frac{\partial P}{\partial \tau} = 0$$

And the boundary conditions.

$$\overline{P}_3\Big|_{r=r_f} = \left(\frac{R_0^2 \rho_0 g^2}{\beta_1 \mu \lambda_\theta \lambda_z} - \frac{\alpha_0 R_0^2}{\beta_1}\right)\frac{\partial^2 P}{\partial \xi^2} + \frac{\beta_1 \mu H}{R_0^2} u_3^* ,$$
$$V_r^{*(3)}\Big|_{r=r_f} = -g\frac{\partial u_3^*}{\partial \xi} + \left(\frac{R_0^2}{2g^3\rho_f^2 \beta_1 \mu H} - \frac{2\beta_2 R_0^3}{g\rho_f \beta_1^3 \mu^2 H^2}\right) P^2 \frac{\partial P}{\partial \xi} \quad (24)$$

From the solution of the sets (23) and (24) we obtain that.

$$\overline{P}_3 = -\frac{r^2}{4}\frac{\partial^2 P}{\partial \xi^2}, \quad (25)$$

$$V_r^{*(3)} = \frac{r^3}{16 g\rho_f}\frac{\partial^3 P}{\partial \xi^3} - \frac{3r}{4g^5\rho_f^3}P^2\frac{\partial P}{\partial \xi} - \frac{r}{g\rho_f}\frac{\partial P}{\partial \tau}, \quad (26)$$

Eliminating the third order perturbed quantities $\overline{P}_3$ and $V_r^{*(3)}$ the desired modified Korteweg-de Vries equation obtained as follows:

$$\frac{\partial P}{\partial \tau} + A\,P^2\frac{\partial P}{\partial \xi} + B\,\frac{\partial^3 P}{\partial \xi^3} = 0 \quad (27)$$

Where the coefficients A and B are defined by.

$$A = \left(\frac{R_0^2}{2g^2\rho_f r_f \beta_1 \mu H} - \frac{2\beta_2 R_0^3}{\beta_1^3 \mu^2 H^2 r_f} + \frac{3}{4g^4\rho_f^2}\right),$$
$$B = \left(\frac{g^2 \rho_f R_0^2 r_f}{4\beta_1 \mu H} + \frac{\rho_0 \rho_f g^4 R_0^4}{\lambda_\theta \lambda_z \beta_1^2 \mu^2 H r_f} - \frac{\alpha_0 g^2 R_0^4 \rho_f}{\beta_1^2 \mu H r_f} - \frac{r_f^2}{16}\right)$$

Our system can support two kinds of potential structure depending on the sign of the coefficient of the nonlinear term (A). A stationary solitary wave solution of the mKdV equation can be obtained by transforming the space variable to:

$$\eta = (\xi - \nu\tau) \quad (28)$$

where ν is velocity of the wave. This has been done by imposing the boundary conditions for localized perturbations, viz., $P=0$, $\frac{dP}{d\eta}=0$, $\frac{d^2P}{d\eta^2}=0$ and for $\eta \to \pm\infty$. Thus, the steady state solution of (27) can be expressed as

$$P = P_0 \sec h\left[\frac{\eta}{\Delta}\right] \quad (29)$$

where the soliton amplitude P_0 and the soliton width Δ are given by

$$P_0 = \sqrt{\frac{6\nu}{A}}, \quad \Delta = \sqrt{\frac{B}{\nu}} \quad (30)$$

NUMERICAL RESULTS AND DISCUSSION

The weakly nonlinear pressure waves in a fluid-filled elastic tube have been investigated. To make our result physically relevant, numerical studies have been made using parameters close to those values corresponding to actual biologically relevant parameters for experimental data in dogs[6]. The effect of the final inner radius of the tube r_f and the wave velocity ν on the basic properties of the amplitude and width of the pres-

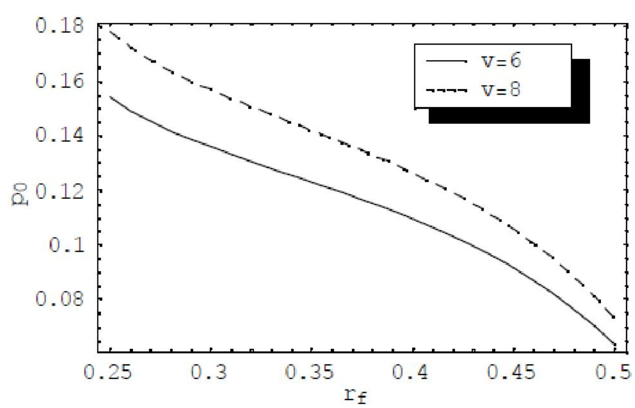

Figure 1 : The variation of the soliton amplitude *P0* with respect to r_f for different values of μ, for different values of ν for β1=296.105, β2=991.496, P_f=1.05 gm/cm³, R_0=0.38 cm, P_0=1.03 gm/cm³, $\lambda_z=\lambda_\theta$=1.6, α_0=78.692, H=2×10⁻² **cm, μ=0.4.**

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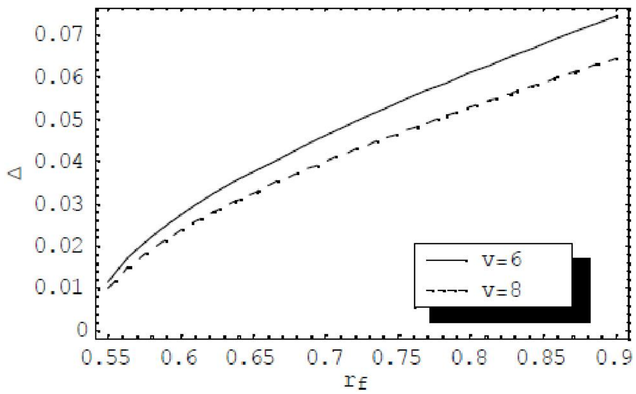

Figure 2 : The variation of the soliton width Δ with respect to r_f for different values of μ, for different values of ν for β_1=296.105, β_2=991.496, P_f=1.05 gm/cm^3, R_0=0.38 cm, P_0 =1.03 gm/cm^3, $\lambda_z=\lambda_\theta$=1.6, α_0=78.692, H=2×10^{-2} cm, μ=0.4.

sure soliton are shown in Figures 1-2. It is obvious from Figures1-2 that the magnitude of the soliton amplitude decrease with the increase r_f and increase with the increase of ν while the width increases with the increase r_f and decrease with the increase of ν. The application of our model might be particularly interesting in the new observations for the biological experimental data.

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