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Determination of periodic solution for a conservative nonlinear singular oscillator in plasma physics

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Abstract

In this paper, an analytical technique has been developed to determine approximate solutions of nonlinear singular oscillator. Usually, a set of nonlinear algebraic equations is solved in this method. However, analytical solutions of these algebraic equations are not always possible, especially in the case of a large oscillation. In this article the small parameter based power series solution to these nonlinear algebraic equations gives the desired results even for the large oscillation. Besides these, a suitable truncation formula is applied to solve this nonlinear singular oscillator which gives the better results than existing solutions. The method is mainly illustrated by the conservative singular oscillator but it is also useful for many other nonlinear problems.

Key Words

Singular oscillator; Approximate solutions; Harmonic balance method; Perturbation.

INTRODUCTION

Many complex real world problems in nature are due to nonlinear phenomena. Nonlinear processes are one of the biggest challenges and not easy to control because the nonlinear characteristic of the system abruptly changes due to some small changes of valid parameters including time. Thus the issue becomes more complicated and hence needs ultimate solution. Therefore, the studies of approximate solutions of nonlinear differential equations (NDEs) play a crucial role to understand the internal mechanism of nonlinear phenomena. One of the most important branches is plasma physic. The discussed equation occurs in the modeling of certain phenomena in plasma physic[28]. Advance nonlinear techniques are significant to solve inherent nonlinear problems, particularly those involving differential equations, dynamical systems and related areas. In recent years, both the mathematicians and physicists have made significant improvement in finding a new mathematical tool would be related to nonlinear differential equations and dynamical systems whose understanding will rely not only on analytic techniques but also on numerical and asymptotic methods. They establish many effective and powerful methods to handle the NDEs.

The study of given nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essential nonlinear and are described by nonlinear equations. It is very difficult to solve nonlinear problems and in general it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. There are many analytical approaches to solve nonlinear differential equations. One of the widely used techniques is perturbation[1-4], whereby the solution is expanded in powers of a small parameter. However, for the nonlinear conservative systems, generalizations of some of the standard perturbation techniques overcome this limitation. In particular, generalization of LP method and He's homotopy perturbation method yield desired results for strongly nonlinear oscillators[5-11].

The harmonic balance method (HBM)[12-22] is another technique for solving strongly nonlinear systems. Usually, a set of difficult nonlinear algebraic equations appears when HBM is formulated. In article[22], such nonlinear algebraic equations are solved in powers of a small parameter. Sometimes, higher approximations also fail to measure the desired results when $a_0 >> 1$. In this article this limitation is removed. Approximate solutions of the same equations are found in which the nonlinear algebraic equations are solved by a new parameter. The higher order approximations (mainly third approximation) have been obtained for mentioned nonlinear oscillator. However, a suitable truncation of these algebraic equations takes the solution very close to the previous one but it saves a lot of calculation. This is the main advantage of the method presented in this article.

THE METHOD

Let us consider a nonlinear differential equation

$$\ddot{x} + \omega_0^2 x = -\varepsilon f(x, \dot{x}), \quad [x(0) = a_0, \dot{x}(0) = 0] \tag{1}$$

where $f(x, \dot{x})$ is a nonlinear function such that $f(-x, -\dot{x}) = -f(x, \dot{x})$, $\omega_0 \geq 0$ and ε is a constant.

Consider a periodic solution of Eq. (1) is in the form

$$x = a_0(\rho\cos(\omega t) + u\cos(3\omega t) + v\cos(5\omega t) + w\cos(7\omega t) + z\cos(9\omega t)\cdots) \tag{2}$$

where a_0, ρ and ω^2 are constants. If $\rho = 1 - u - v - \cdots$ and the initial phase $\varphi_0 = 0$, solution Eq. (2) readily satisfies the initial conditions $[x(0) = a_0, \dot{x}(0) = 0]$.

Substituting Eq. (2) into Eq. (1) and expanding $f(x, \dot{x})$ in a Fourier series, it converts to an algebraic identity

$$\begin{aligned} &a_0[\rho(\omega_0^2 - \omega^2)\cos(\omega t) + u(\omega_0^2 - 9\omega^2)\cos(3\omega t) + \cdots] = \\ &-\varepsilon[F_1(a_0, u, \cdots)\cos(\omega t) + F_3(a_0, u, \cdots)\cos(3\omega t) + \cdots] \end{aligned} \tag{3}$$

By comparing the coefficients of equal harmonics of Eq. (3), the following nonlinear algebraic equations are found

$$\begin{aligned} &\rho(\omega_0^2 - \omega^2) = -\varepsilon F_1, \quad u(\omega_0^2 - 9\omega^2) = -\varepsilon F_3, \\ &v(\omega_0^2 - 25\omega^2) = -\varepsilon F_5, \cdots \end{aligned} \tag{4}$$

With the help of the first equation, ω^2 is eliminated from all the rest of Eq. (4). Thus Eq. (4) takes the following form

$$\begin{aligned} &\rho\omega^2 = \rho\omega_0^2 + \varepsilon F_1, \quad 8\omega_0^2 u\rho = \varepsilon(\rho F_3 - 9uF_1), \\ &24\omega_0^2 v\rho = \varepsilon(\rho F_5 - 25vF_1), \cdots \end{aligned} \tag{5}$$

Substitution $\rho = 1 - u - v - \cdots$, and simplification, second-, third- equations of Eq. (5) take the following form

$$\begin{aligned} &u = G_1(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \\ &v = G_2(\omega_0, \varepsilon, a_0, u, v, \cdots, \lambda_0), \cdots \end{aligned} \tag{6}$$

where G_1, G_2, $\cdots$ exclude respectively the linear terms of u, v, $\cdots$.

Whatever the values of ω_0 and a_0, there exists a parameter $\lambda_0(\omega_0, \varepsilon, a_0) << 1$, such that u, v, $\cdots$ are expandable in following power series in terms of λ_0 as

$$u = U_1\lambda_0 + U_2\lambda_0^2 + \cdots, \quad v = V_1\lambda_0 + V_2\lambda_0^2 + \cdots, \quad \cdots \tag{7}$$

where U_1, U_2, $\cdots$, V_1, V_2, $\cdots$ are constants.

Finally, substituting the values of u, v, $\cdots$ from Eq. (7) into the first equation of Eq. (5), ω is determined. This completes the determination of all related functions for the proposed periodic solution as given in Eq. (2).

EXAMPLES

Let us consider an important and interesting nonlinear differential equation which was studied in plasma physic[28]

$$\ddot{x} + x^{-1} = 0 \tag{8}$$

The Eq. (8) can be written as

$$x\ddot{x} + 1 = 0 \tag{9}$$

We consider the first-order approximate solution of Eq. (9) is

$$x(t) = a_0(\cos(\omega_1 t) \tag{10}$$

Now substituting Eq. (10) into the Eq. (9) it takes the following form

$$\frac{1}{a_0} - \frac{a_0\omega_1^2}{2} - \frac{a_0\omega_1^2\cos(2\omega_1 t)}{2} = 0 \tag{11}$$

Equating the constant term equal to zero the first approximate angular frequency is

$$\frac{1}{a_0} - \frac{a_0\omega_1^2}{2} = 0$$

$$i.e.\ \omega_1 = \sqrt{\frac{2}{a_0^2}} = \frac{1.41421}{a_0} \tag{12}$$

Therefore the first-order approximation solution of Eq. (9) is Eq. (10) where ω_1 is given by Eq. (12).

We use the solution of the form of Eq. (2) a second-order approximate solution of Eq. (9) is

$$x(t) = a_0(\rho\cos(\omega_2 t) + u\cos(3\omega_2 t) \tag{13}$$

Substituting $\rho = 1 - u$, along with Eq. (13) into the Eq. (9) and then setting the coefficients of constant term and $\cos(2\omega_2 t)$, the following nonlinear algebraic equations are obtained

$$\frac{1}{a_0} - \frac{a_0\omega_2^2}{2} + a_0\omega_2^2 u - 5a_0\omega_2^2 u^2 = 0 \tag{14}$$

$$-\frac{a_0\omega_2^2}{2} - 4a_0\omega_2^2 u + \frac{9a_0\omega_2^2 u^2}{2} = 0 \tag{15}$$

After simplification, Eq. (14) takes the form

$$\omega_2^2 = \frac{2}{a_0^2 - 2a_0^2 u + 10a_0^2 u^2} \tag{16}$$

By elimination of ω_2^2 from Eq. (15), the equation of

u is obtained as

$$u = \lambda_0\left(-1+9u^2\right) \quad \lambda_0 = \frac{1}{8} \tag{17}$$

The power series solution of Eq. (17) is

$$u = -\lambda_0 + 9\lambda_0^3 - 162\lambda_0^5 + 3645\lambda_0^7 + \cdots . \tag{18}$$

Substituting the value of u from Eq. (18) into the Eq. (16) the second-order approximate angular frequency is

$$\omega_2 = \sqrt{\frac{2}{a_0^2 - 2a_0^2u + 10a_0^2u^2}} = \frac{1.22004}{a_0} \tag{19}$$

Thus the second-order approximation solution of Eq. (9) is $x(t) = a_0(\rho\cos(\omega_2 t) + u\cos(3\omega_2 t)$ where u and ω_2 are respectively given by Eqs. (18) and (19).

It is observed that solution Eq. (13) measures better result when Eqs. (14)-(15) is truncated as

$$\frac{1}{a_0} - \frac{a_0\omega_2^2}{2} + a_0\omega_2^2u - 5a_0\omega_2^2u^2/2 = 0 \tag{20}$$

$$-\frac{a_0\omega_2^2}{2} - 4a_0\omega_2^2u + \frac{9a_0\omega_2^2u^2}{4} = 0 \tag{21}$$

Seeing that Eq. (14)-(15), it is clear that the half of the second order terms are considered. Now from the Eq. (19) we can easily obtain

$$\omega_2^2 = \frac{2}{a_0^2 - 2a_0^2u + 5a_0^2u^2} \tag{22}$$

Combining the Eq. (22) and Eq. (21) and then simplifying we get

$$u = \lambda_0\left(-2+9u^2\right), \qquad \lambda_0 = \frac{1}{16} \tag{23}$$

The power series solution of Eq. (23) is

$$u = -2\lambda_0 + 36\lambda_0^3 - 1296\lambda_0^5 + 58320\lambda_0^7 + \cdots \tag{24}$$

Substituting the value of u from Eq. (24) into the Eq. (22) the angular frequency in truncation form is

$$\omega_{trun2} = \sqrt{\frac{2}{a_0^2 - 2a_0^2u + 5a_0^2u^2}} = \frac{1.23884}{a_0} \tag{25}$$

In a similar way, the method can be used to determine higher order approximations. In this article, a third-order approximate solution of Eq. (9) is

$$\begin{aligned} x(t) &= a_0\cos(\omega_3 t) + a_0u(\cos(3\omega_3 t) - \cos(\omega_3 t) \\ &+ a_0v(\cos(5\omega_3 t) - \cos(\omega_3 t) \end{aligned} \tag{26}$$

Substituting Eq. (26) into the Eq. (9) and equating the constant term and the coefficients of $\cos(2\omega_3 t)$ and $\cos(4\omega_3 t)$ the following nonlinear algebraic equations are found

$$\begin{aligned} &1 - a_0^2\omega_3^2/2 + a_0^2\omega_3^2u - 5a_0^2\omega_3^2u^2 \\ &+ a_0^2\omega_3^2v - a_0^2\omega_3^2uv - 13a_0^2\omega_3^2v^2 = 0 \end{aligned} \tag{27}$$

$$\begin{aligned} &-a_0^2\omega_3^2/2 - 4a_0^2\omega_3^2u + 9a_0^2\omega_3^2u^2/2 \\ &+ a_0^2\omega_3^2v - 13a_0^2\omega_3^2uv - a_0^2\omega_3^2v^2/2 = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} &-5a_0^2\omega_3^2u + 5a_0^2\omega_3^2u^2 - 13a_0^2\omega_3^2v \\ &+ 18a_0^2\omega_3^2uv + 13a_0^2\omega_3^2v^2 = 0 \end{aligned} \tag{29}$$

From the Eq. (27) we can easily written as

$$\omega_3^2 = \frac{1}{a_0^2/2 - a_0^2u + 5a_0^2u^2 - a_0^2v + a_0^2uv + 13a_0^2v^2} \tag{30}$$

Now using Eq. (30) into the Eq. (28)-(29) we get the equation of u and v are

$$u = \lambda_0\left(-1 + 9u^2 + 2v - 26uv - v^2\right) \tag{31}$$

$$v = \mu_0(-5u + 5u^2 + 18uv + 13v^2) \tag{32}$$

where λ_0 is defined in the Eq. (17) and $\mu_0 = \frac{1}{13}$.

The algebraic relation between λ_0 and μ_0 is

$$\mu_0 = 8\lambda_0/13 \tag{33}$$

The power series solutions of u and v from the Eqs. (31)-(32) in terms of λ_0 are

$$\begin{aligned} u &= -\lambda_0 + 197\lambda_0^3/13 + 1120\lambda_0^4/ \\ &13 - 61458\lambda_0^5/169 - 71680\lambda_0^6/13 + \cdots \end{aligned} \tag{34}$$

$$\begin{aligned} v &= 40\lambda_0^2/13 + 40\lambda_0^3/13 - 13640\lambda_0^4/169 - 53520\lambda_0^5/ \\ &169 + 4725200\lambda_0^6/2197 + 50472200\lambda_0^7/2197 + \cdots \end{aligned} \tag{35}$$

Substituting the values of u and v from Eq. (34)-(35) into Eq. (30), simplifying the third-order approximate angular frequency is

$$\omega_3 = \sqrt{\frac{1}{a_0^2/2 - a_0^2u + 5a_0^2u^2 - a_0^2v + a_0^2uv + 13a_0^2v^2}} = \frac{1.250302}{a_0} \tag{36}$$

Therefore a third-order approximation periodic solution of Eq. (9) is define as Eq. (26) where u, v and ω_3 are respectively given by the Eqs. (34)-(36).

The third-order approximate solution Eq. (26) measures almost similar result when Eqs. (27)-(29) are truncated as

$$1 - a_0^2\omega_3^2/2 + a_0^2\omega_3^2u - 5a_0^2\omega_3^2u^2 + a_0^2\omega_3^2v - a_0^2\omega_3^2uv/2 = 0 \tag{37}$$

$$\begin{aligned} &-a_0^2\omega_3^2/2 - 4a_0^2\omega_3^2u + 9a_0^2\omega_3^2u^2/2 + a_0^2\omega_3^2v \\ &-13a_0^2\omega_3^2uv/2 = 0 \end{aligned} \tag{38}$$

$$-5a_0^2\omega_3^2u + 5a_0^2\omega_3^2u^2 - 13a_0^2\omega_3^2v + 18a_0^2\omega_3^2uv/2 = 0 \tag{39}$$

Seeing that Eqs. (37)-(39), it is clear that the higher order terms of u and v (more than third) are ignored; but half of the third order terms are considered.

From the Eq. (37) we can easily written as

$$\omega_3^2 = \frac{1}{a_0^2/2 - a_0^2u + 5a_0^2u^2 - a_0^2v + a_0^2uv/2} \tag{40}$$

Now using the Eq. (40) into the Eq. (37)-(38) we get the equation of u and v are

$$u = \lambda_0\left(-1 + 9u^2 + 2v - 13uv\right) \tag{41}$$

$$v = \mu_0(-5u + 5u^2 + 9uv) \tag{42}$$

Where λ_0 and μ_0 are defined as the Eq. (17) and Eq. (32).

The algebraic relation between λ_0 and μ_0 is obtained as Eq. (33).

The power series solutions of u and v from the Eqs. (41)-(42) in terms of λ_0 are

$$\mathbf{u} = -\lambda_0 + 197\lambda_0^3/13 + 600\lambda_0^4/13 - 60858\lambda_0^5/169 - 36000\lambda_0^6/13 + \cdots \tag{43}$$

$$\mathbf{v} = 40\lambda_0^2/13 + 40\lambda_0^3/13 - 10760\lambda_0^4/169 - 3280\lambda_0^5/13 + \cdots \tag{44}$$

Substituting the values of u and v from Eq. (43)-(44) into Eq. (40), simplifying the third-order approximate angular frequency in truncation form is

$$\omega_{\text{trun3}} = \sqrt{\frac{1}{a_0^2/2 - a_0^2 u + 5a_0^2 u^2 - a_0^2 v + a_0^2 uv/2}} = \frac{1.260817}{a_0} \tag{45}$$

Therefore a third-order approximation periodic solution of Eq. (8) is define as Eq. (26) where u, v and ω_3 are respectively given by the Eqs. (43)-(45).

RESULTS AND DISCUSSIONS

We illustrate the accuracy of the simple analytical method by comparing the approximate angular frequencies previously obtained with the exact one. For this nonlinear problem, the exact angular frequency is[26]

$$\omega_e(a_0) = \frac{1.253314}{a_0} \tag{46}$$

The approximate angular frequencies and their relatives errors (*RE*) obtained in this paper by applying a simple analytical technique to this nonlinear singular oscillator are the following

$$\omega_1(a_0) = \frac{1.41421}{a_0} \quad \text{RE} = 12.8\%$$

$$\omega_{\text{trun2}}(a_0) = \frac{1.23884}{a_0} \quad \text{RE} = 1.1\% \tag{47}$$

$$\omega_{\text{trun3}}(a_0) = \frac{1.260817}{a_0} \quad \text{RE} = 0.6\%$$

Where the percentage errors (*RE*) were calculated using the following equation

$$\text{RE} = 100 \times \left| \frac{\omega_i(a_0) - \omega_e(a_0)}{\omega_e(a_0)} \right| \qquad i = 1, 2, 3 \tag{48}$$

Mickens, R. E.[26], approximately solved Eq. (8) using Harmonic balance method (HBM). They achieved the following results for the first and second-order approximate angular frequency are as follows

$$\omega_1(a_0) = \frac{1.41421}{a_0} \quad \text{RE} = 12.8\% \tag{49}$$

$$\omega_2(a_0) = \frac{1.272792}{a_0} \quad \text{RE} = 1.6\% \tag{50}$$

Mickens, R., E.[27], approximately solved Eq. (8) using an improved harmonic balance method that incorporates the salient features of both Newton's method and the harmonic balance method. They calculated the following results for the first and second-order approximate angular frequency in orders

$$\omega_1(a_0) = \frac{1.41421}{a_0} \quad \text{RE} = 12.8\% \tag{51}$$

$$\omega_2(a_0) = \frac{1.219113}{a_0} \quad \text{RE} = 2.7\% \tag{52}$$

Comparing all the approximates results to their corresponding numerical values, we observe that the first-order approximate angular frequency is same in all method but the second-order approximate angular frequency obtained in this paper by truncation principle is better than those obtained in Mickens, R., E.[26,27]. Using truncation principle it safe a lot of calculation compared with without truncation principle. It has been mentioned that the procedure of Mickens, R., E.[26,27], is laborious especially for obtaining the higher approximations. The advantages of this method include its simplicity and computational efficiency, and the ability to objectively better agreement in third-order approximate solution.

CONCLUSION

Based on a truncation principle of the related algebraic equations in HBM, a new analytical technique has been presented to determine approximate periodic solutions of nonlinear singular oscillator. In compared with the previously published methods, determination of solutions is straightforward and simple. And also we see that the approximate angular frequency in second- approximate solution using the truncation principle the relative error is 1.1% whereas the article those obtained Mickens, R., E.[26,27], are 1.6% and 2.7%. To sum up we can say that the method presented in this article for solving nonlinear singular oscillator can be considered as an efficient alternative of the previously proposed methods.

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