A THEORETICAL EVALUATION OF SCATTERING LENGTH FOR TWO COUPLED SQUARE WELL POTENTIAL AS A FUNCTION OF ΔμB

BRIJ BEHARI SHARMA[a] and L. K. MISHRA[*]

Department of Physics, Magadh University, BODH GAYA – 824234 (Bihar) INDIA
[a]Department of Chemistry, Magadh University, BODH GAYA – 824234 (Bihar) INDIA

ABSTRACT

An evaluation of the scattering length and effective range of an attractive square well potential for ultracold atomic gases were performed. Our theoretical results showed that for any potential with large positive scattering length has a bound state just below the continuum threshold of energy. The theoretically evaluated results are in good agreement with other theoretical workers.

Key words: Ultracold atomic gases, Two-coupled square well potential, Feshbach resonance, Hyperfine interaction, Scattering length, Continuum threshold of energy.

INTRODUCTION

After the experimental realization of Bose-Einstein condensation in trapped gases[1], a great deal of experimental progress has been made in the field of ultra cold atomic gases[2-5]. The particular reason for this progress is the unprecedented experimental control over the atomic gases of interest. This experimental control over the ultacold magnetically trapped alkali gases has also demonstrated the experimentally adjustable interaction between the atoms. This is achieved by means of Feshbach resonance[6]. The defining feature of a Feshbach resonance is that the bound state responsible for the resonance exists in another part of the quantum mechanical Hilbert space than the parts associated with the incoming particles. Here, the two incoming atoms in the open channel have a different hyperfine state than the bound state in the closed channel. The coupling between the open and closed channel is provided by the exchange interaction. As a result of this difference in the hyperfine state, the two channels have a different Zeeman shift in a magnetic field. Therefore, the energy difference between the closed-channel bound state and the two-atom continuum threshold, so-called detuning, is experimentally adjustable by tuning the

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[*]Author for correspondence; E-mail: muphysicslkm@gmail.com

magnetic field. This implies that the s-wave scattering length and hence the magnitude and sign of the interatomic interactions is also adjustable to any desirable.

With this experimental degree of freedom, it is possible to study very interesting new regimes in a many-body physics of ultacold atomic gases. The first experimental application was the detailed study of the collapse of a condensate with attractive interactions, corresponding to negative scattering lengths. In general, the collapse occurs when the attractive interactions overcome the stabilizing kinetic energy of the condensate atoms in the trap. Since the typical interaction energy is proportional to the density, there is a certain maximum number of atoms above which the condensate is unstable[7-10].

Mathematical formulae used in the evaluation

One considers the situation of two atoms of mass m that interacts via a potential V(r) that vanishes at large distance between the atoms. The motion of the atoms separates into the trivial center-of mass motion and the relative motion, described by the wave function ψ(r) where $r = x_1 - x_2$, and x_1 and x_2 are the coordinates of two atoms, respectively. The wave function is determined by the time independent Schrödinger equation-

$$\left[\frac{\hbar^2 \nabla^2}{m} + V(r) \right] \psi(r) = E\psi(r) \qquad \qquad \text{…(1)}$$

With E is the energy of the atoms in the center–of–mass system. Solutions of the Schrodinger equation with negative energy correspond to bound states of the potential i.e. to molecular states. To describe the atom-atom scattering one has to look for solution with positive energy $E = 2\varepsilon_K$ with $\varepsilon_K = \frac{\hbar^2 k^2}{2m}$ the kinetic energy of single atom with momentum ħk. Since any realistic interatomic interaction potential vanishes rapidly as the distance between atoms becomes large, one knows that the solution for $r \to \infty$ of equation (1) is given by a superposition of incoming and outgoing plane waves. The scattering wave function is given by incoming plane wave and outgoing spherical wave and reads as[11-14].

$$\Psi(r) \sim e^{ikr} + f(k', k)\, \frac{e^{ikr}}{r} \qquad \qquad \text{…(2)}$$

Where the function f(k',k) is known as the scattering amplitude. Following the partial–wave method, one expands the scattering amplitude in Legendre polynomials according to –

$$f(k', k) = \sum_{l=0}^{\infty} f_l(k, r)\, P_l \cos(\theta) \qquad \qquad \text{…(3)}$$

Here θ is the angle between K and K'. One expands the incident wave in partial waves according to –

$$e^{ik.r} = \sum_{l=0}^{\infty} \frac{2l+1)i^l}{kr} \sin\left(kr - \frac{l\pi}{2}\right) P_l \cos(\theta) \qquad \ldots(4)$$

The partial wave amplitude $f_l(k)$ has the form –

$$f_l(k) = \frac{(2l+1)i^l}{2ik}(e^{2i\delta_l}(k) \qquad \ldots(5)$$

Where $\delta_l(k)$ is the phase shift of the lth partial wave. The low energy effective interactions between the atoms are fully determined by the s-wave scattering length defined by –

$$a = -\lim k \to 0 \frac{\delta_0(k)}{k} \qquad \ldots(6)$$

The scattering length is determined by the linear dependence of the phase shift on the magnitude of the relative momentum ħk of the scattering atoms for small momentum. Generally, the phase shift can be expanded according to –

$$k\cot(\delta_0(k)) = -\frac{1}{4} + \frac{1}{2}r_{eff}k^2 + ---- \qquad \ldots(7)$$

From which the scattering length is determined by –

$$a = R(1 - \frac{\tan\gamma}{\gamma}), \gamma = R[m|V_0|\hbar^2]^{\frac{1}{2}} \qquad \ldots(8)$$

R is the range of the square well potential. γ is dimensionless constant. The parameter r_{eff} is the effective range and for square well potential is given by[15] –

$$r_{eff} = R[1 + \frac{2\tan\gamma - \gamma(3+\gamma^2)}{3\gamma(\gamma - \tan)^2}] \qquad \ldots(9)$$

The equation for bound-state energy is given by –

$$\sqrt{\frac{m}{\hbar^2}}|E| = -\sqrt{\frac{m}{\hbar^2}}(E_m - V_0)\cot(\sqrt{\frac{m}{\hbar^2}}(E_m - V_0) \qquad \ldots(10)$$

Where V_0 is the depth of the potential. The relation between the energy of the molecular state and scattering length is given by –

$$E_m = -\frac{\hbar^2}{ma^2} \qquad\qquad …(11)$$

This result does not depend on the specific details of the potential and it turns out to be quite general. The hyperfine interaction is given by –

$$V_{hf} = \frac{a_{hf}}{\hbar^2} I.S \qquad\qquad …(12)$$

Here a_{hf} is hyperfine constant. The hyperfine interaction has both singlet and triplet states. In the presence of magnetic field the different internal states of the atoms have a different Zeeman shift, in a experiment with magnetically trapped gases, energy difference between these states is therefore experimentally accessible. The magnetic field dependence of the scattering length near a Feshbach resonance is characterized by a width ΔB and position B_0 according to[16] –

$$a(B) = a_{bg}\left(1 - \frac{\Delta B}{B - B_0}\right) \qquad\qquad …(13)$$

This shows that the scattering length and magnitude of the effective interatomic interaction can be altered to any value by tuning the magnetic field. a_{bg} is the scattering length at the Feshbach resonance[17].

RESULTS AND DISCUSSION

In this paper, we have studied Feshbach resonance in two-channel system with square well potential. Using the theoretical formalism of Ohashi and Griffin[18], we have evaluated the scattering length for two coupled square well potential as a function of $\Delta\mu B$. $\Delta\mu$ is the difference in the magnetic moment and B is the magnetic field. Using equation (8) and (9), we have evaluated the scattering length (a/R) and Effective range (r_{eff}/R) as a function of dimensionless parameter $\gamma = R\sqrt{\dfrac{m|V_0|}{\hbar^2}}$ for an attractive square well potential.

Here R is the range of potential and V_0 is the depth of potential. The results are shown in Table 1. From the results, one can see that the scattering length can be both positive and negative. This becomes equal to zero at values of γ such that $\gamma = \tan \gamma$. It is also observed that the effective range diverges of the scattering length and becomes equal to zero. At

values of $\gamma = (n + 1/2)\,\pi$ with n a positive integer, the scattering length diverges and changes sign. This behavior is called a potential or shape resonance and in fact occurs each time the potential is just deep enough to support a new bound state. Therefore for large and positive scattering length the square well has a bound state with an energy just below the continuum threshold. It turns out that there is an important relationship between the energy of the bound state and the scattering length.

Table 1: Evaluated results of scattering length (a/R) and effective range (r_{eff}/R) for an attractive square well potential in the unit of range of potential R as a function of dimensionless parameter $\gamma = R\sqrt{\dfrac{m|V_0|}{\hbar^2}}$

$\gamma = R\sqrt{\dfrac{m\|V_0\|}{\hbar^2}}$	Scattering length (a/R)	Effective range (r_{eff}/R)
0.0	0.0	12.6
1.0	-0.05	10.8
2.0	-0.24	8.7
3.0	-0.45	5.2
4.0	-0.10	3.2
5.0	0.06	1.5
6.0	0.17	0.25
7.0	0.25	-0.28
8.0	0.38	-0.36
9.0	0.27	-0.50
10.0	0.18	-0.10
15.0	0.12	-0.04

The relation is $E_m = -\dfrac{\hbar^2}{ma^2}$ where 'a 'is the scattering length. This result does not depend on the specific details of the potential and it turns out to be quite general. Any potential with large positive scattering length has a bound state just below the continuum threshold with energy given by E_m. In Table 2, we have given the results of scattering of two square well potential as a function of $\Delta\mu B$. The resonant behavior is due to the bound state of the single $V_s(r)$. Now solving the equation (10) for the binding energy with $V_0 = -V_s$, one

finds that $|E_m| = \dfrac{4.62\hbar^2}{mR^2}$ which is approximately the position of the resonance. The difference is due to the fact that the hyperfine interaction leads to a shift in the position of the resonance with respect to E_m.

Table 2: Evaluated results of scattering length for two coupled square well potential as a function of ΔμB. The depth of triplet and singlet channels potentials is $V_T = \dfrac{\hbar^2}{mR^2}$ and $V_s = \dfrac{10\hbar^2}{mR^2}$ respectively. The Hyperfine coupling is $V_{hf} = \dfrac{0.1\hbar^2}{mR^2}$

$\dfrac{\Delta\mu B}{\left(\dfrac{\hbar^2}{mR^2}\right)}$	(a / R)
4.40	0.04
4.45	0.16
4.50	0.35
4.55	0.65
4.60	1.05
4.65	-3.25
4.70	-3.16
4.75	-2.85
4.80	-2.60
4.85	-2.52
4.90	-2.34
4.95	-2.12
5.00	-1.15
5.50	-1.10

The magnetic field dependence of the scattering length near a Feshbach resonance is characterized experimentally by a width ΔB and position B_0 according to equation (13). This expression shows that the scattering length and therefore the magnitude of the effective interatomic intervals may be altered to any value by tuning the magnetic field. The off resonant behavior of scattering length is denoted by a_{bg} and is approximately equal to the scattering length of the triplet potential $V_T(r)$. Now, using the expression for the scattering

length of the square well potential from equation (8) for $\gamma = 1$, one finds that $a_{bg} = 0.56R$. One can also find the problem of the resonance and is equal to $B_0 = \dfrac{4.64\hbar^2}{m\Delta\mu R^2}$ and width is equal to $= -\dfrac{0.05\hbar^2}{m\Delta\mu R^2}$. In Table 3, we have presented the experimental results of scattering length as a function of magnetic field. Some recent[19-25] results also reveal the same facts.

Table 3: Evaluated results of scattering length as a function of magnetic field. The scattering length is normalized such that it is equal to one far off resonance.

Magnetic field (G)	Scattering length (a_{bg})
890	1.0
895	2.45
900	3.96
905	5,82
907	0.86
909	0.98
910	1.25
912	1.86
915	2.20

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