ISSN : 0974 - 7435 Volume 8 Issue 1

BioTechnology
An Indian Journal

FULL PAPER

BTAIJ, 8(1), 2013 [1-10]

Study of RLT-enhanced and lifted formulations for the job-shop scheduling problem

Yonghui Cao[1,2]
[1]School of Economics & Management, Henan Institute of Science and Technology, (CHINA)
[2]School of Management, Zhejiang University, (CHINA)
E-mail: caoyonghui2000@126.com

ABSTRACT

In this paper, we propose novel continuous nonconvex as well as lifted discrete formulations of the notoriously challenging class of job-shop scheduling problems with the objective of minimizing the maximum completion time. In particular, we develop an RLT-enhanced continuous nonconvex model for the job-shop problem based on a quadratic formulation of the job sequencing constraints on machines. The tight linear programming relaxation that is induced by this formulation is then embedded in a globally convergent branch-and-bound algorithm. Furthermore, we design another novel formulation for the job-shop scheduling problem that possesses a tight continuous relaxation, where the non-overlapping job sequencing constraints on machines are modeled via a lifted asymmetric traveling salesman problem (ATSP) construct, and specific sets of valid inequalities and RLT-based enhancements are incorporated to further tighten the resulting mathematical program.

KEYWORDS

Reformulation-linearization technique;
Lifted formulations;
Job-shop scheduling formulations.

INTRODUCTION

The deterministic job-shop scheduling problem (JSSP) arises in many industrial environments and presents a classical combinatorial optimization problem that has proven to be highly challenging to solve. The extent of research conducted in this field over the last forty years or so has motivated several surveys, such as the survey by Blazewicz et al.[1] and the state-of-the-art review by Jain and Meeran[2]. The computational intractability of this problem is illustrated by the fact that the 10-job-machine test problem FT10, introduced by Fisher and Thompson[3] in 1963, was provably solved to optimality for the first time by Carlier and Pinson[4] more than two decades later in 1989. Although several mathematical programming formulations have been proposed for the JSSP since the late fifties, little progress has been realized with this trend of research, principally because of the weakness of the underlying continuous relaxations of the formulated models and the tremendous consequent computational effort required to solve the associated pure or mixed-integer programs. Reflecting on the difficulty of the JSSP, Conway et al.[5] observed, quite emphatically, that: “Although it is easy to state, and to visualize what is required, it is extremely difficult to make any progress whatever toward a solution. Many proficient people have considered the problem, and all have come away essentially empty-handed."

However, recent developments in solving mixed-integer programs together with modern computer capabilities resurrect some hope in this direction and, with this motivation, we investigate in this chapter several new modeling and lifting concepts for the JSSP with the objective of minimizing the maximum completion time.

In the second part, we introduce our notation along with Manne's model for the deterministic JSSP. In the third part, we propose an enhanced continuous nonconvex mathematical program for this problem using the RLT methodology, and investigate an RLT-based Lagrangian dual formulation that is further enhanced via semidefinite cuts. The fourth part delineates and discusses a globally convergent optimization algorithm where RLT formulations play a key role in providing tighter relaxations. In The fifth part, we propose enhanced LP relaxations for the JSSP based on a novel formulation in which the non-overlapping job sequencing constraints on machines are modeled via a lifted asymmetric traveling salesman problem (ATSP) viewpoint, and various sets of valid inequalities and RLT-lifted constraints are proposed to further tighten the resulting representation.

NOTATION AND SOME EARLY MODELS

Several mathematical programming formulations have been proposed for the JSSP. These early works are reviewed in detail in Appendix A, but we focus here on the most popular and useful model due to Manne[6], as well as certain nonlinear, nonconvex modifications suggested by Nepomiastchy[7] and Rogers[8], which will be exploited using new modeling concepts and RLT-based enhancements discussed later in this paper.

Notation

Below is a summary of our notation.

- M =set of m machines.
- J =set of n jobs.
- J_j = set of ordered operations of job j.
- Dummy operation 0 that marks the start (and the end) of all operation sequences on all machines.
- $J_0 = J \cup \{0\}$
- F^* =set of first operations, that is, the first operation of each job is included in this set.
- F_i^* =subset of F^* that is to be performed on machine i, $\forall i$.
- E^* =set of last operations, that is, the last operation of each job is included in this set.
- E_i^* = subset ofthat is to be performed on machine,.
- *T*=operation of job j to be performed on machine i.
- p_{ij}=processing time of O_{ij}
- $A_j=\{(i_1j,i_2j)$: Operationis O_{i1j} required to immediately precede operation O_{i2j} of job $j\}$ =set of conjunctive arcs that represent precedence constraints between (ordered) operations belonging to job j.
- $D_i=\{(ij_1,ij_2)$: both jobs j_1 and j_2, $j_1<j_2$, require operations O_{ij1} and O_{ij2} to be performed on machine i in a disjunctive *fashion*$\}$
- $P(O_{ij})$=set of all operations of job j that precede operation O_{ij}
- $S(O_{ij})$=set of all operations of job j that follow O_{ij}
- *T*= upper bound on the makespan.
- $[l_{ij}, u_{ij}]$=time interval for commencing operation O_{ij}. Such lower and upper bounds can be computed by setting $l_{ij} = \sum_{k:O_{kj} \in P(O_{ij})} p_{kj}$ and $u_{ij} = T - (p_{kj} + \sum_{k:O_{kj} \in P(O_{ij})} p_{kj})$
- T_i={set of triplets of distinct job indices (j_1, j_2, j_3) such that it is possible to perform the respective operations of these jobs in this order on machine, $\forall i \in M$.

Manne's Model (1960)

For convenience, and because of the popularity of this formulation, we state Manne's model for the JSSP, and refer the interested reader to Appendix A for a detailed chronological account of alternative existing formulations in the literature.

Decision variables

- t_{ij}=starting time of
- $z^i_{j_1j_2} = \begin{cases} 1 & \text{if operation } j_1 \text{ is performed sometime prior to operation } j_2 \text{ on machine } i \\ 0 & \text{others}, \forall (ij_1, ij_2) \in D_i, i \in M. \end{cases}$
- $C_{\max} = \max\{t_{ij} + p_{ij} : O_{ij} \in E^*\}$. $C_{\max}$ is the makespan or the maximum completion time of a schedule.

Minimize $C_{\max}$ **(1a)**

Subject to $C_{\max} \geq t_{ij} + p_{ij}, \forall O_{ij} \in E^*$ **(1b)**

$t_{i_2 j} - t_{i_1 j} \geq p_{i_1 j}, \forall j \in J, (i_1 j, i_2 j) \in A_j$ **(1c)**

$\varphi \geq 0 \varphi_{i j_1 j_2} \cdot \varphi_{i j_2 j_1} = 0, \forall j_1 \neq j_2 \in J, i \in M)(t_{i j_2} - t_{i j_1} - p_{ij_1}) \leq 0, \forall i \in M, (ij_1, ij_2) \in D_i$ **(1d)**

$t_{i j_1} - t_{i j_2} + K z^i_{j_1 j_2} \geq p_{ij_2}, \forall i \in M, (ij_1, ij_2) \in D_i$ **(1e)**

$z\ binary, t \geq 0$ **(1f)**

where K is a suitably large number. The objective function (1a) and Constraint (1b) express the objective of minimizing the maximum completion time. Constraint (1c) enforces the precedence restrictions between (ordered) operations that belong to job j, whereas Constraints (1d)-(1e) model the non-overlapping job sequencing constraints on machines via disjunctive relationships. Constraint (1f) enforces logical binary and nonnegativity restrictions on the problem variables.

This model provides the most compact formulation among early models for the JSSP, and was used in Greenberg's[9] B&B algorithm for the job-shop problem, as well as in Balas'[10] application of a specialized version of the filter method to the JSSP. Instead of the discrete non-overlapping job sequencing constraints (1d-1e) utilized in Manne's model, Nepomiastchy suggested the following nonlinear, continuous, nonconvex constraints:
$(t_{i j_1} - t_{i j_2} - p_{ij_2})(t_{i j_2} - t_{i j_1} - p_{ij_1}) \leq 0, \forall i \in M, (ij_1, ij_2) \in D_i$

The problem was then tackled using a penalty function approach that could terminate at a local, possibly non-global, optimum. In a similar spirit, Rogers adopted the following linear-quadratic constraints to model the foregoing disjunctive relationships:

$\varphi_{i j_1 j_2} - t_{i j_1} + t_{i j_2} \geq p_{ij_1}, \forall j_1 \neq j_2 \in J, i \in M$ **(2a)**

$\varphi_{i j_2 j_1} - t_{i j_2} + t_{i j_1} \geq p_{ij_2}, \forall j_1 \neq j_2 \in J, i \in M$ **(2b)**

$\varphi_{i j_1 j_2} \cdot \varphi_{i j_2 j_1} = 0, \forall j_1 \neq j_2 \in J, i \in M$ **(2c)**

$\varphi \geq 0$ **(2d)**

Here, the role of the binary variables used in Manne's model is played by the complementarity constraints (2d). Again, a local search procedure was proposed to tackle this nonlinear, nonconvex formulation.

Valid Inequalities in the literature

Valid inequalities, or cutting planes, are frequently adopted to strengthen the continuous relaxations of combinatorial optimization problems. The main task here is to formulate classes of valid inequalities that not only tighten the model representation and help significantly improve its continuous relaxation-based lower bound, but also can be generated efficiently within a reasonable amount of time. Ideally, it is desirable to generate valid inequalities that characterize facets of the convex hull of feasible solutions to the MIP problem, but judiciously generated strong cutting planes or lifted versions of model-defining constraints can also greatly enhance the computational performance.

Applegate and Cook[11] offer an interesting analysis of the effect of valid inequalities on lower bounds for both disjunctive and MIP formulations of the JSSP. Their study includes newly developed valid inequalities as well as those proposed by Balas[12] and Dyer and Wolsey[13]. We identify below certain key valid inequalities that have been proposed in the literature in order to strengthen the underlying LP relaxations of Manne's model.

- Basic cuts (attributed to Dyer and Wolsey[13] in [11]):
  $\sum_{j \in C} p_{ij} t_{ij} \geq \min_{j \in C} l_{ij} \sum_{j \in C} p_{ij} + \sum_{j_1, j_2 \in C: j_1 < j_2} p_{ij_1} p_{ij_2}, \forall C \subseteq J, \forall i \in M$
- Half cuts[11]:
  $t_{ij_1} \geq \min_{j \in C} l_{ij} + \sum_{j_2 \in C, j_2 < j_1} z_{ij_2 j_1} p_{ij_2} + \sum_{j_2 \in C, j_1 < j_2} (1 - z_{ij_1 j_2}) p_{ij_2}, \forall j_1 \in J, C \subseteq J, i \in M$
- Basic cuts plus epsilon[11]:
  $\sum_{j \in C} p_{ij} t_{ij} \geq l_{ik} \sum_{j \in C} p_{ij} + \sum_{j_1, j_2 \in C, j_1 < j_2} p_{ij_1} p_{ij_2} - \left(\sum_{j \in C, j < k} z_{ijk} \left\{l_{ik} - l_{ij}\right\}^+ + \sum_{j \in C, j < k} (1 - z_{ijk}) \left\{l_{ik} - l_{ij}\right\}^+\right) \sum_{j \in C} p_{ij},$
  $\forall k \in J, C \subseteq J, i \in M, where \left\{l_{ik} - l_{ij}\right\}^+ = \max\left\{0, l_{ik} - l_{ij}\right\}$
- Triangle cuts[11]:
  $z_{ij_1 j_2} + z_{ij_2 j_3} + z_{ij_1 j_3} \leq 1, \forall j_1 < j_2 < j_3 \in J, i \in M$

RLT-BASED CONTINUOUS MODEL AND LINEAR LOWER BOUNDING PROBLEM

RLT-based relaxation

Adopting the continuous nonconvex disjunctive constraints suggested by Nepomiastchy, we can reformulate Manne's[14] model as follows, where we have defined a new variable $g^i_{j_1 j_2}$ to represent the difference $t_{ij_1} - t_{ij_2}$, $\forall i \in M, \forall i \in J, \forall (ij_1, ij_2) \in D_i$, for the sake of analytical convenience.

Minimize C Max **(3a)**

Subject to $C \max \geq t_{ij} + p_{ij}, \forall j \in E^*_i, i \in M$ **(3b)**

$t_{i_2 j} - t_{i_1 j} \geq p_{i_1 j}, \forall j \in J, \forall (i_1 j, i_2 j) \in A_j$ **(3c)**

$(g^i_{j_1 j_2} - p_{ij_2})(g^i_{j_1 j_2} + p_{ij_1}) \geq 0, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(3d)**

$g^i_{j_1 j_2} = t_{ij_1} - t_{ij_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(3e)**

$t \geq 0$ **(3f)**

Based on the order of operations O_{ij1} and O_{ij2} in

FULL PAPER

the routing of jobs j_1 and j_2, we can derive lower and upper bounds on the starting time of any operation O_{ij} of the type $l_{ij} = \sum_{k:O_{kj} \in P(O_{ij})} p_{kj}$ and $u_{ij} = T - \sum_{k:O_{kj} \in S(O_{ij})} p_{kj} - p_{ij}$, where T is some upper bound on the optimal makespan. Although T may be computed via any adequate heuristic, efficient algorithms (such as the Shifting Bottleneck procedure) should be preferred, because the tightness of the bounds on the variables significantly contributes to the strength of the constraints generated by RLT constructs. Thus, we deduce box-constraints of the type $\alpha^i_{j_1j_2} \le g^i_{j_1j_2} \le \beta^i_{j_1j_2}$, where $\alpha^i_{j_1j_2} = l_{ij_1} - u_{ij_2}$ and $\beta^i_{j_1j_2} = u_{ij_1} - l_{ij_2}$. Using these bounding constraints, we can augment the job-shop formulation with the following RLT bounding-factor product relationships, denoted $F^i_{j_1j_2} \ge 0$, of the type: $(\beta^i_{j_1j_2} - g^i_{j_1j_2})(g^i_{j_1j_2} - \alpha^i_{j_1j_2}) \ge 0$, $(\beta^i_{j_1j_2} - g^i_{j_1j_2})^2 \ge 0$, and $(g^i_{j_1j_2} - \alpha^i_{j_1j_2})^2 \ge 0, \quad \forall i \in M, \forall (j_1, j_2) \in D_i$. Hence, we derive the following Manne-based RLT-enhanced formulation, which we denote by JQP.

JQP: Minimize $C_{\max}$ **(4a)**

Subject to $C\max \ge t_{ij} + p_{ij}, \forall j \in E_i^*, i \in M$ **(4b)**

$t_{i_2j} - t_{i_1j} \ge p_{i_1j}, \forall j \in J, \forall (i_1j, i_2j) \in A_j$ **(4c)**

$(g^i_{j_1j_2} - p_{ij_2})(g^i_{j_1j_2} + p_{ij_1}) \ge 0, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(4d)**

$g^i_{j_1j_2} = t_{ij_1} - t_{ij_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(4e)**

$F^i_{j_1j_2} \ge 0, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(4f)**

$t \ge 0$ **(4g)**

As per the RLT methodology, this reformulated problem can be linearized to yield a lower bounding linear program by using the following RLT variable substitution identities:

$$h^i_{j_1j_2} = \left[g^i_{j_1j_2}\right]^2, \forall i \in M, \forall (ij_1, ij_2) \in D_i \quad \textbf{(5)}$$

Denoting $[.]_l$ by the operator that linearizes polynomial functions under (5), we have that:

$$\left[F^i_{j_1j_2}\right]_l \ge 0 \Leftrightarrow \begin{cases} h^i_{j_1j_2} \le g^i_{j_1j_2}(\alpha^i_{j_1j_2} + \beta^i_{j_1j_2}) - \alpha^i_{j_1j_2}\beta^i_{j_1j_2} \\ h^i_{j_1j_2} \ge 2\beta^i_{j_1j_2} g^i_{j_1j_2} - (\beta^i_{j_1j_2})^2 \\ h^i_{j_1j_2} \ge 2\alpha^i_{j_1j_2} g^i_{j_1j_2} - (\alpha^i_{j_1j_2})^2 \end{cases}$$

The RLT-based linear programming relaxation, JLP, is thus obtained as given below.

JLP: Minimize $C_{\max}$ **(6a)**

Subject to $C\max \ge t_{ij} + p_{ij}, \forall j \in E_i^*, i \in M$ **(6b)**

$t_{i_2j} - t_{i_1j} \ge p_{i_1j}, \forall j \in J, \forall (i_1j, i_2j) \in A_j$ **(6c)**

$h^i_{j_1j_2} \ge (p_{ij_2} - p_{ij_1}) g^i_{j_1j_2} + p_{ij_1}p_{ij_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(6d)**

$g^i_{j_1j_2} = t_{ij_1} - t_{ij_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(6e)**

$h^i_{j_1j_2} \le g^i_{j_1j_2}(\alpha^i_{j_1j_2} + \beta^i_{j_1j_2}) - \alpha^i_{j_1j_2}\beta^i_{j_1j_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(6f)**

$h^i_{j_1j_2} \ge 2\beta^i_{j_1j_2} g^i_{j_1j_2} - (\beta^i_{j_1j_2})^2, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(6g)**

$h^i_{j_1j_2} \ge 2\alpha^i_{j_1j_2} g^i_{j_1j_2} - (\alpha^i_{j_1j_2})^2, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(6h)**

$t, h \ge 0$ **(6i)**

As obvious from the foregoing derivation, JLP is indeed a lower bounding problem for JQP. Moreover, if the substitution identities (5) are satisfied in an optimal solution to JLP, then this solution also yields an optimum for JQP.

Remark 1. Observe that the bounding constraints of the type $\alpha^i_{j_1j_2} \le g^i_{j_1j_2} \le \beta^i_{j_1j_2}$ are not explicitly enforced in JLP. In fact, the bound-factors $g^i_{j_1j_2} - \alpha^i_{j_1j_2} \ge 0$ and $\beta^i_{j_1j_2} - g^i_{j_1j_2} \ge 0$ are dominated by the higher order bound-factor products inherent within $\left[F^i_{j_1j_2}\right]_l \ge 0$.

Remark 2. Note that $\alpha^i_{j_1j_2} \ge 0$ implies that O_{ij2} must precede O_{ij1}. Similarly, if $\beta^i_{j_1j_2} \le 0$, then O_{ij1} must precede O_{ij2}. Therefore, once any disjunction is resolved, i.e. $\alpha^i_{j_1j_2} \ge 0$ or $\beta^i_{j_1j_2} \le 0$ is determined, we replace the associated constraints in (6d)-(6h) by $g^i_{j_1j_2} \ge p_{ij_2}$ or $g^i_{j_1j_2} \le -p_{ij_1}$, respectively.

Lagrangian dual formulations

In this section, we investigate a basic Lagrangian dual relaxation that is further enhanced via semidefinite cuts in order to tighten the model formulation (see Sherali and Fraticelli[15] and Sherali and Desai[16]).

Basic formulation

Denoting Lagrange multipliers λ_{ij} associated with (6b), for $j \in E_i^*, i \in M$, $\pi_{j,(i_1j,i_2j)}$ associated with (6c) for $j \in J$, $(i_1j, i_2j) \in A_j$, $\mu_{ij_1j_2}$ associated with

(6d), $\phi_{ij_1j_2}$ associated with (6g), and $\delta_{ij_1j_2}$ associated with (6h) for $i \in M$, $(ij_1, ij_2) \in D_i$ we can formulate the following Lagrangian Dual to JLP, which we denote by JLD1.

DLD1:

$$Maximize\left\{\theta(\lambda,\pi,\mu,\phi,\delta): \sum_{i\in M}\sum_{j\in E_i^*}\lambda_{ij} = 1, \lambda > 0, \pi > 0, \mu > 0, \phi > 0, \delta > 0\right\} \quad \textbf{(7)}$$

where

$$\begin{aligned}\theta(\lambda,\pi,\mu,\phi,\delta) = Minimum\Big\{&\sum_{i\in M}\sum_{j\in E_i^*}\lambda_{ij}(t_{ij}+p_{ij}),\\ &+\sum_{j=1}^{n}\sum_{(i_1j,i_2j)\in A_j}\pi_{j,(i_1j,i_2j)}(p_{i_1j}-t_{i_2j}+t_{i_1j})\\ &+\sum_{i=1}^{m}\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}\mu_{ij_1j_2}(p_{ij_1}p_{ij_2}+(p_{ij_2}-p_{ij_1})g^i_{j_1j_2}-h^i_{j_1j_2})\\ &+\sum_{i=1}^{m}\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}\phi_{ij_1j_2}(2\beta^i_{j_1j_2}g^i_{j_1j_2}-(\beta^i_{j_1j_2})^2-h^i_{j_1j_2})\\ &+\sum_{i=1}^{m}\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}\delta_{ij_1j_2}(2\alpha^i_{j_1j_2}g^i_{j_1j_2}-(\alpha^i_{j_1j_2})^2-h^i_{j_1j_2})\Big\}\end{aligned}$$

(8a)

subject to $g^i_{j_1j_2} = t_{ij_1} - t_{ij_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$ **(8b)**

$$0 \le h^i_{j_1j_2} \le g^i_{j_1j_2}(\alpha^i_{j_1j_2}+\beta^i_{j_1j_2}) - \alpha^i_{j_1j_2}\beta^i_{j_1j_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i \quad \textbf{(8c)}$$

$$l_{ij} \le t_{ij} \le u_{ij}, \forall i \in M, j \in J \quad \textbf{(8d)}$$

and where we have imposed the implied bounds on the t-variables in the subproblem constraints (8d) in order to ensure a finite optimum for this problem. For convenience, we shall denote the objective function expression in (8a) as "Obj (8a)".

SDP-enhanced formulation

JLD1 can be further enhanced by incorporating a class of SDP-based constraints in the spirit of the SDP cuts introduced by Sherali and Fraticelli. To this end, we consider the vector $g(1) = \begin{bmatrix}1\\ g^i_{j_1j_2}\end{bmatrix}$, and define the following matrix $H^i_{j_1j_2} \equiv \left[g(1)g^T_{(1)}\right]_l = \begin{bmatrix}1 & g^i_{j_1j_2}\\ g^i_{j_1j_2} & h^i_{j_1j_2}\end{bmatrix}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$. Requiring $H^i_{j_1j_2}$ to be positive semidefinite, that is $H^i_{j_1j_2} \succ 0$, we enforce constraints of the type $h^i_{j_1j_2} \ge (g^i_{j_1j_2})^2, \forall i \in M, \forall (ij_1, ij_2) \in D_i$,. This leads to the following RLT-based, SDP-enhanced, Lagrangian dual formulation, JLD2, where the Lagrangian multipliers associated with the dualization of (6e) are denoted $\eta_{ij_1j_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i$.

JLD2:

$$Maximize\left\{\theta'(\lambda,\pi,\mu,\phi,\delta,\eta): \sum_{i\in M}\sum_{j\in E_i^*}\lambda_{ij} = 1, \lambda > 0, \pi > 0, \mu > 0, \phi > 0, \delta > 0, \eta\ unrestricted\right\} \quad \textbf{(9)}$$

where

$$\theta'(\lambda,\pi,\mu,\phi,\delta) == Minimum\left\{Obj(8a) + \sum_{i=1}^{m}\sum_{j_1=1}^{n-1}\sum_{j_2=j_1+1}^{n}\eta_{ij_1j_2}(g^i_{j_1j_2} - t_{ij_1} + t_{ij_2})\right\} \quad \textbf{(10a)}$$

subject to

$$(g^i_{j_1j_2})^2 \le h^i_{j_1j_2} \le g^i_{j_1j_2}(\alpha^i_{j_1j_2}+\beta^i_{j_1j_2}) - \alpha^i_{j_1j_2}\beta^i_{j_1j_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i \quad \textbf{(10b)}$$

$$l_{ij} \le t_{ij} \le u_{ij}, \forall i \in M, j \in J \quad \textbf{(10c)}$$

$$\alpha^i_{j_1j_2} \le g^i_{j_1j_2} \le \beta^i_{j_1j_2}, \forall i \in M, \forall (ij_1, ij_2) \in D_i \quad \textbf{(10d)}$$

Deflected subgradient optimization techniques are worthy of exploration in order to solve JLD1 and JLD2. Specialized efficient schemes for evaluating the Lagrangian dual objective functions shall be developed in this research. Observe that the objective coefficients pertaining to the h-variables in (10a) are nonpositive and, therefore, the upper bound on the -variables represented in (10b) will be binding due to the minimization operation. Hence, we shall also investigate an alternative strategy in which the upper bounding expression in (10b) is dualized and accommodated within (10a), while requiring $(g^i_{j_1j_2})^2 = h^i_{j_1j_2}$ in lieu of (10b). Note that the latter constraint can be equivalently replaced by the convex hull of this restriction and (10d). We shall exploit this structure in designing efficient schemes for optimizing such Lagrangian dual formulations.

BRANCH-AND-BOUND ALGORITHM

In this section, we present a globally convergent B&B algorithm in concert with the RLT-based formulation.

Let Ω denote the hyperrectangle bounding the *g*-variables at the root node of the B&B search tree, and accordingly, let us denote the original problem and its corresponding lower bounding problem as JQP (Ω) and JLP (Ω), respectively. Likewise, for any subnode *k*, we define the sub-hyperrectangle $\Omega^k \subseteq \Omega$ and the corresponding problems JQP (Ω^k) and JLP (Ω^k). Let ν[.] be the value at optimality of any given problem [.]. For convenience, we also denote the vector of t_{ij}-variables by, and similarly we introduce the vectors *g* and *h*.

If at any node k in the B&B tree, the optimal solution $(\bar{t}, \bar{g}, \bar{h})$ obtained for JLP (Ω^k) satisfies the variable substitution identities (5), then $(\bar{t}, \bar{g})$ solves JQP (Ω^k). That is, all the RLT variables faithfully reproduce the squared variables they represent, and a feasible solution to the original problem is thereby available that achieves the lower bounding value. As a consequence, the incumbent solution and its value for the original problem, (t^*, g^*)and C max*, can potentially be updated as necessary. Also, if (5) holds for JLP (Ω) at the root node of the search tree, then the solution obtained to JLP (Ω) is indeed optimal to JQP (Ω), and the algorithm terminates.

As noted in Remark 2, $g^i_{j_1j_2} \geq 0 \Rightarrow g^i_{j_1j_2} \geq p_{ij_2}$, and similarly $g^i_{j_1j_2} \leq 0 \Rightarrow g^i_{j_1j_2} \leq -p_{ij_1}$. That is, imposing one sign or another to any variable $g^i_{j_1j_2}$ is equivalent to a binary decision that fixes the relative order of operations O_{ij1} and O_{ij2} on machine *i*. This result is at the heart of the branching rule.

Branching rule

Consider some node k in the search tree. The partitioning step is based on the identification of the variable $g^i_{j_1j_2}$ that creates the highest discrepancy between an RLT variable and the term it replaces. We select $g^i_{j_1j_2}$ such that

$(i, j_1j_2) \in \underset{i\in M, (ij_1, ij_2)\in D_i}{\arg\max} \left\{\rho^i_{j_1j_2}\right\}$, where

$$\rho^i_{j_1j_2} = \left|h^i_{j_1j_2} - (g^i_{j_1j_2})^2\right|$$

Upon the $g^i_{j_1j_2}$ selection of , we create two new nodes by partitioning Ω^k into $\Omega^{k+1} \equiv \Omega^k \cap \left\{g^i_{j_1j_2} \geq p_{ij_2}\right\}$ and $\Omega^{k+2} \equiv \Omega^k \cap \left\{g^i_{j_1j_2} \leq -p_{ij_1}\right\}$.

A formal description of the overall B&B algorithm is given below.

Step 0: Initialization Step. Initialize the incumbent solution (t^*, g^*) and its objective value by computing a heuristic solution. (We used the SBP [1] for this purpose.) Set k=1 and $\Omega^k = \Omega$. Solve JLP (Ω) and denote its optimal solution by $(\bar{t}, \bar{g}, \bar{h})$. Determine a branching variable $g^i_{j_1j_2}$ according to the branching rule. If $\rho^i_{j_1j_2} = 0$, then $(\bar{t}, \bar{g})$ is optimal to JQP; terminate the algorithm after setting $g(t^*, g^*) \leftarrow (\bar{t}, \bar{g})$, and $C\max^* \leftarrow v[JLP(\Omega)]$. Otherwise, if $|\rho^i_{j_1j_2}| > 0$, proceed to Step 1, with the selected node $\hat{k} = 1$.

Step 1: Branching Step. Create two new nodes, (k + 1) and (k +2), by partitioning $\Omega^{\hat{k}}$, into Ω^{k+1} and as explained above, and remove the parent node, , from the list of active nodes.

Step 2: Bounding Step. Solve JLP (Ω^{k+1}) and JLP (Ω^{k+2}). Update the incumbent if appropriate. Select and store a branching variable for each of these two nodes. Increment $k \leftarrow k+2$.

Step 3: Fathoming Step. Fathom any node k such that $v[\text{JLP}(\Omega^k)] \geq C\max^*(1-\varepsilon)$ by removing it from the list of active nodes, where $0 \leq \varepsilon \leq 1$ is a specified percentage optimality gap (use $\varepsilon = 0$ if a global optimal is desired). If the list of active nodes is empty, stop. Otherwise, proceed to Step 4.

Step 4: Node Selection Step. Among the active nodes, select one ($\hat{k}$, say) that has the least lower bound, and go to Step 1.

Proposition 1. The foregoing B&B algorithm (run with $\varepsilon = 0$) terminates finitely and produces an optimal solution to JQP at termination.

Proof. The result directly follows from the branching strategy because there are a finite number of ways of resolving the disjunctions.

Note that the deepest level that can be reached in the search tree is $\frac{mn(n-1)}{2}$, which corresponds to fixing the signs of $g^i_{j_1j_2}$, $\forall i \in M, \forall(ij_1, ij_2) \in D_i$.

LIFTED ATSP-BASED FORMULATIONS

In a recent paper, Sherali et al.[17] have proposed several lifting concepts and RLT-enhancements for the ATSP with and without precedence constraints, and have demonstrated the tightness of the resulting formulations for various standard benchmark problems. In the context of the JSSP, we shall adopt an insightful modeling approach where the scheduling of jobs to be performed on any machine is viewed as an ATSP problem, and certain sets of valid inequalities and RLT-enhancements are derived, as established in the sequel below.

Decision Variables

- t_{ij}=starting time of
- $x^i_{j_1 j_2} = \begin{cases} 1 \text{ if the operation of job } j_1 \text{ immediately precedes the operation of } j_2 \text{ on machine } i \\ 0 \text{ otherwise, } \forall j_1 \neq j_2 \in J_0, i \in M \end{cases}$
- $y^i_{j_1 j_2} = \begin{cases} 1 \text{ if the operation of job } j_1 \text{ performed sometime prior the operation of job } j_2 \text{ on machine } i \\ 0 \text{ otherwise, } \forall j_1 \neq j_2 \in J_0, i \in M \end{cases}$
- $C\max = \max\left\{t_{ij} + p_{ij} : O_{ij} \in E^*\right\}$. *C max* is the makespan or the maximum completion time of a schedule.

JS-ATSP1:*Minimize C max* **(11a)**

subject to $C\max \geq t_{ij_1} + p_{ij_1} + \sum_{j_2 \neq j_1} p_{ij_2} y^i_{j_1 j_2}, \forall j_1 \in E^*_i, i \in M$ **(11b)**

$$\sum_{j_2 \in J_0 - \{j_1\}} x^i_{j_1 j_2} = 1, \forall j_1 \in J_0, i \in M \qquad \textbf{(11c)}$$

$$\sum_{j_1 \in J_0 - \{j_2\}} x^i_{j_1 j_2} = 1, \forall j_2 \in J_0, i \in M \qquad \textbf{(11d)}$$

$$y^i_{j_1 j_2} + y^i_{j_2 j_1} = 1, \forall j_1 < j_2 \in J, i \in M \qquad \textbf{(11e)}$$

$$y^i_{j_1 j_2} \geq x^i_{0 j_1}, \forall j_1 \neq j_2 \in J, i \in M \qquad \textbf{(11f)}$$

$$y^i_{j_1 j_2} \geq x^i_{j_1 0}, \forall j_1 \neq j_2 \in J, i \in M \qquad \textbf{(11g)}$$

$$y^i_{j_1 j_2} \geq x^i_{j_1 j_2}, \forall j_1 \neq j_2 \in J, i \in M \qquad \textbf{(11h)}$$

$$y^i_{j_1 j_3} \geq (y^i_{j_1 j_2} + y^i_{j_2 j_3} - 1) + x^i_{j_2 j_1}, \forall (j_1, j_2, j_3) \in \Gamma_i, i \in M \qquad \textbf{(11i)}$$

$$t_{ij_2} \geq t_{ij_1} + p_{ij_1} - (1 - y^i_{j_1 j_2})(p_{ij_1} + u_{ij_1} - l_{ij_2}), \forall j_1 \neq j_2 \in J, i \in M \qquad \textbf{(11j)}$$

$$t_{ij_2} \geq t_{ij_1} + p_{ij_1} + \sum_{\substack{j \neq j_1 \\ j \neq j_2}} x^i_{j_1 j} p_{ij} - (1 - y^i_{j_1 j_2})(p_{ij_1} + u_{ij_1} - l_{ij_2} + \max_{j \neq j_1, j_2}\{p_{ij}\}), \forall j_1 \neq j_2 \in J, i \in M \qquad \textbf{(11k)}$$

$$t_{i_2 j} - t_{i_1 j} \geq p_{i_1 j}, \forall j \in J, (i_1 j, i_2 j) \in A_j \qquad \textbf{(11l)}$$

$$\sum_{i \in M} \sum_{j_1 \in F^*_i} x^i_{0 j_1} \geq 1 \qquad \textbf{(11m)}$$

$$t_{ij_2} \geq \sum_{j_1 \in J - \{j_2\}} y^i_{j_1 j_2} p_{ij_1}, \forall j_2 \in J, i \in M \qquad \textbf{(11n)}$$

$$t_{ij_1} \leq T - \sum_{j_2 \in J - \{j_1\}} y^i_{j_1 j_2} p_{ij_2}, \forall j_1 \in J, i \in M \qquad \textbf{(11o)}$$

$$l_{ij} \leq t_{ij} \leq u_{ij}, \forall j \in J, i \in M \qquad \textbf{(11p)}$$

$$x \text{ binary}, y \geq 0 \qquad \textbf{(11q)}$$

The objective function (11a), in conjunction with Constraint (11b), enforces the definition of the makespan as the maximum completion time of the schedule. Observe that Constraint (11b) provides a lifted expression of the makespan constraint formulated in Manne's model, taking into consideration the completion time of the last operation of every job and augmenting this with the sum of the processing times of the operations scheduled after it. For the remainder of the formulation, in essence, we exploit the analogy between the set of job-operations to be performed on any machine, augmented with a dummy node 0, and the cities to be visited in an ATSP given the base city 0, in order to sequence the operations assigned to this machine via Constraints (11c)-(11i) and (11q). Constraint (11j) computes the start-times of operations on each machine given the y-variables and is partially lifted via Constraint (11k) as established in Proposition 2 below. Constraint (11l) enforces the precedence relationships among operations that belong to the same job. Constraint (11m) ensures that, for at least one machine, call it *i*, the first operation to be processed must belong to F^*_i. The bounds in Constraints (11n)-(11p) are determined by examining the relative position of any operation, O_{ij}, in the sequence of operations to be processed on machine and in the sequence of operations that belong to job *j*. Constraint (4.11q) enforces logical binary restrictions on the -variables and the nonnegativity of the -variables. Observe that the binariness of the -variables together with Constraints (11e), (11h), and (11i) induce binary restrictions on the -variables.

Proposition 2. Constraints (11k) enforce a set of valid inequalities.

Proof.

If $y^i_{j_1 j_2} = 1$, then $t_{ij_2} \geq t_{ij_1} + p_{ij_1} + \sum_{\substack{j \neq j_1 \\ j \neq j_2}} x^i_{j_1 j} p_{ij}$, which is valid since job j_1 precedes (not necessarily immediately) job j_2 on machine *i*. On the other hand, if $y^i_{j_1 j_2} = 0$, then $t_{ij_2} - l_{ij_2} \geq t_{ij_1} - u_{ij_1} + \sum_{\substack{j \neq j_1 \\ j \neq j_2}} x^i_{j_1 j} p_{ij} - \max_{j \neq j_1, j_2}\{p_{ij}\}$, which is valid $t_{ij_1} - u_{ij_1} \leq 0$ since and $\sum_{\substack{j \neq j_1 \\ j \neq j_2}} x^i_{j_1 j} p_{ij} - \max_{j \neq j_1, j_2}\{p_{ij}\} \leq 0$, while $t_{ij_2} - l_{ij_2} \geq 0$.

There are two sets of optional, alternative valid inequalities that can be investigated in the context of JS-ATSP1 based on the formulations developed by Sherali et al. The first of these replaces Constraint (11i) by the following:

$$y^i_{j_1 j_3} \geq (x^i_{j_1 j_2} + y^i_{j_2 j_3} - 1) + (x^i_{j_1 j_3} + x^i_{j_2 j_3}), \forall (j_1, j_2, j_3) \in \Gamma_i, i \in M \qquad \textbf{(12)}$$

Hence, we obtain the following job-shop model:

JS-ATSP2:

Minimize {Cmax : (11b)-(11q), with (4.12) enforced in lieu of (11i)}.(4.13)

Sherali et al. also describe certain RLT-lifted constraints for the ATSP that are predicated on defining the following product variables in the present context:

$$f^i_{j_1 j_3 j_2} = x^i_{j_1 j_3} y^i_{j_3 j_2} \forall (j_1, j_3, j_2) \in \Gamma_i, i \in M \qquad \textbf{(14)}$$

FULL PAPER

These variables are then related to the original - and -variables in JS-ATSP1 via the following valid inequalities:

$$\sum_{j_3:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} + x^i_{j_1j_2} = y^i_{j_1j_2}, \forall j_1 \neq j_2 \in E_i, i \in M \quad \textbf{(15a)}$$

$$\sum_{j_1:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} + x^i_{0j_3} = y^i_{j_3j_2}, \forall j_3 \neq j_2 \in E_i, i \in M \quad \textbf{(15b)}$$

$$0 \le f^i_{j_1j_3j_2} \le x^i_{j_1j_3}, \forall (j_1, j_3, j_2) \in \Gamma_i, i \in M \quad \textbf{(15c)}$$

Proposition 3. (a) Constraints (15a)-(15c) are valid and (b) Constraints (15a)-(15c) along with Problem (4.11) guarantee that the RLT-based Constraint (14) hold true.

Proof.

(a) Observe that Constraint (15c) is trivially valid by the binariness of the - and -variables and the definition of the -variables in Constraint (14).

The validity of Constraint (15a) is established next by distinguishing three cases:

- If $x^i_{j_1j_2} = 1$, then $y^i_{j_1j_2} = 1$ and $x^i_{j_1j_3} = 0$, $\forall j_3 \neq j_1, j_2$. Hence $\left| f^i_{j_1j_3j_2} = 0, \forall j_3 \neq j_1, j_2\right.$, by Constraint (14) and, therefore, Constraint (15a) holds true.
- If $x^i_{j_1j_2} = 0 \wedge y^i_{j_1j_2} = 1$, then there must exist a unique job j for which $\sum_{j_3:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} = x^i_{j_1j} y^i_{jj_2} = y^i_{j_1j_2} = 1$. Hence, Constraint (15a) is valid.
- If $x^i_{j_1j_2} = 0 \wedge y^i_{j_1j_2} = 0$, then job j_2 precedes j_1 on machine i, and it must be that $f^i_{j_1j_3j_2} = x^i_{j_1j_3} y^i_{j_2j_3} = 0$, $\forall j_3 \neq j_1, j_2$,

Likewise, the validity of Constraint (15b) is established below by considering three cases:

- If $x^i_{0j_3} = 1$, then $y^i_{j_3j_2} = 1$ and $x^i_{j_1j_3} = f^i_{j_1j_3j_2} = 0$, $\forall j_3 \neq j_1, j_2$, and, hence, Constraint (15b) is valid.
- If $x^i_{0j_3} = 0 \wedge y^i_{j_3j_2} = 1$, then Constraint (15b) equivalently asserts that $\sum_{j_1:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} = \sum_{j_1 \neq j_3} x^i_{j_1j_3} = 1$, which is valid by Constraint (11d).
- If $x^i_{0j_3} = 0 \wedge y^i_{j_3j_2} = 0$, then $f^i_{j_1j_3j_2} = 0$, $\forall j_3 \neq j_1, j_2$ and, hence, Constraint (15b) is valid.

(b) Now, we shall show that Constraints (15b)-(15c) in concert with Problem (4.11) imply the RLT substitution equations in (14).

- If $[x^i_{j_1j_3} = 0 \wedge (y^i_{j_3j_2} = 0 \vee y^i_{j_3j_2} = 1)]$, then $f^i_{j_1j_3j_2} = 0$ by Constraint (15c) and, therefore, Constraint (14) holds true.
- If $x^i_{j_1j_3} = 1 \wedge y^i_{j_3j_2} = 0$, then $x^i_{0j_3} = 0$ and Constraint (15b) implies that $\sum_{j_1:\{j,j_3,j_2\}\in\Gamma_i} f^i_{jj_3j_2} = 0$. Therefore, invoking the nonnegativity of the -variables, we deduce that $f^i_{j_1j_3j_2} = 0$, and Constraint (14) is valid.
- If $x^i_{j_1j_3} = 1 \wedge y^i_{j_3j_2} = 1$, then $x^i_{0j_3} = 0$ and Constraint (15b) implies that $f^i_{j_1j_3j_2} + \sum_{j \neq j_1:\{j,j_3,j_2\}\in\Gamma_i} f^i_{jj_3j_2} = 1$. However, $f^i_{j_1j_3j_2} \le 1$ by (15c) and, since $x^i_{j_1j_3} = 1$, then $x^i_{jj_3} = 0$, $\forall j \neq j_1$, such that $\{j, j_3, j_2\} \in \Gamma_i$ by (11d), and so, $f^i_{jj_3j_2} = 0$, $\forall j \neq j_1$, such that $\{j, j_3, j_2\} \in \Gamma_i$, by (15c). Thus, $f^i_{j_1j_3j_2} = 1$, and Constraint (14) is valid.

Also, under (15a)-(15c), we can lift Constraint (11j) and replace it by the following valid inequality, as proven next in Proposition 4.

$$t_{ij_2} \ge t_{ij_1} + p_{ij_1} + \sum_{j_3:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} p_{ij_3} - (1 - y^i_{j_1j_2})(p_{ij_1} + u_{ij_1} - l_{ij_2}) =, \forall j_1 \neq j_2 \in J, i \in M \quad \textbf{(16)}$$

Proposition 4. Constraint (16) enforces a set of valid inequalities.

Proof. We shall examine three cases to establish the validity of (16):

- If $[y^i_{j_1j_2} = 1 \wedge x^i_{j_1j_2} = 1]$, then job j_1 is the immediate predecessor of job j_2 on machine i, and $\sum_{j_3 \neq j_2} x^i_{j_1j_3} = 0$ by (11c). Hence, $\sum_{j_3:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} p_{ij_3} = 0$, and (16) equivalently asserts that $t_{ij_2} \ge t_{ij_1} + p_{ij_1}$, which is valid.
- If $[y^i_{j_1j_2} = 1 \wedge x^i_{j_1j_2} = 0]$, then job j_1 precedes of job j_2, but is not its immediate predecessor on machine i. Therefore, there exists a unique job j such that $j \neq j_2$ and $x^i_{j_1j} = 1$, and $\sum_{j_3:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} p_{ij_3} = x^i_{j_1j} y^i_{jj_2} p_{ij} = p_{ij}$, Hence, (16) equivalently asserts that $t_{ij_2} \ge t_{ij_1} + p_{ij_1} + p_{ij}$, which is valid.
- If $y^i_{j_1j_2} = 0$, then $\sum_{j_3:\{j_1,j_3,j_2\}\in\Gamma_i} f^i_{j_1j_3j_2} p_{ij_3} = 0$, Hence, (16) equivalently asserts that $t_{ij_2} - l_{ij_2} \ge t_{ij_1} - u_{ij_1}$, which is true because $t_{ij_2} - l_{ij_2} \ge 0$, while $t_{ij_1} - u_{ij_1} \le 0$

We also introduce the following constraints, which are validated by Proposition 5 below:

$$C\max \geq t_{ij} + p_{ij} + p_{ij_1} x^i_{jj_1} + \sum_{j_2 \neq j, j_1} p_{ij_2} f^i_{jj_1j_2}, \forall j \in E^*_i, i \in M \quad (17)$$

Proof.

Observe that Constraint (17) is derived from the lifted makespan constraint formulated in (11b). Now, if $x^i_{jj_1} = 0$, then $f^i_{jj_1j_2} = 0, \forall j_2 \neq j, j_1$, and Constraint (17) reduces to $C\max \geq t_{ij} + p_{ij}$, which is valid. On the other hand, if $x^i_{jj_1} = 1 \Rightarrow f^i_{jj_1j_2} = y^i_{j_1j_2}$, and Constraint (17) reduces to $C\max \geq t_{ij} + p_{ij} + \sum_{j_2 \neq j, j_1} p_{ij_2} f^i_{j_1j_2}$, which is again valid.

Noting that (15a) and (15b) respectively imply (11h) and (11f) under $f \geq 0$, we get the following RLT-lifted alternative formulation of JS-ATSP1.

JS-ATSP3: Minimize {Cmax : (11b)-(11q) and (17),with (11f) and (11h) replaced by (4.15a)-(4.15c), and (11j) replaced by (16)}. (18a)

Remark 3. Similar to the variant JS-ATSP2 derived from JS-ATSP1, we could attempt the following alternative to JS-ATSP3.

JS-ATSP4: Minimize {Cmax : Constraints of JS-ATSP3 where (11i) is replaced by (12)} (18b)

CONCLUSIONS

We have proposed novel continuous nonconvex as well as lifted discrete formulations for the challenging class of job-shop scheduling problems with the objective of minimizing the maximum completion time. More generally in the literature on the benefits of the RLT methodology for minimax and discrete optimization problems, we developed an RLT-enhanced continuous nonconvex model for the job-shop problem based on a quadratic formulation of the job sequencing constraints on machines due to Nepomiastchy. The lifted linear programming relaxation that is induced by this formulation was then embedded in a globally convergent branch-and-bound algorithm. Further more, we designed another novel formulation for the job-shop scheduling problem that possesses a tight continuous relaxation, where the non-overlapping job sequencing constraints on machines are modeled via a lifted asymmetric traveling salesman problem (ATSP) construct, and specific sets of valid inequalities and RLT-based enhancements are incorporated to further tighten the resulting mathematical program. In addition, we suggest that a theoretical investigation of dominance relationships between our ATSP-based formulation and alternative MIP formulations of the JSSP be conducted for future research. We also propose to evaluate the RLT-based Lagrangian dual formulations, and possibly integrate these within our B&B algorithm in lieu of the RLT-based linear programming relaxation to accelerate the computational performance and enhance the B&B pruning effect. Finally, it would be worthwhile to apply the general-purpose lifting procedures for strengthening the JSSP formulation, and compare the induced relaxation against our ATSP-based formulations that were lifted using specialized valid inequalities and RLT constructs.

REFERENCES

[1] J.Blazewicz, W.Domschke, E.Pesch; The job-shop scheduling problem: conventional and new solution techniques, European Journal of Operational Research, **93(1)**, 1-33 August 23 **(1996)**.

[2] A.S.Jain, S.Meeran; Deterministic job-shop scheduling: Past, present and future, European Journal of Operational Research, **113**, 390-434 **(1999)**.

[3] H.Fisher, G.L.Thompson; Probabilistic learning combinations of local jobshop scheduling rules, in J.F.Muth, G.L.Thompson, (Eds); Industrial Scheduling, Prentice-Hall, Englewood Cliffs, NJ, 225-251 **(1963)**.

[4] J.Carlier, E.Pinson; An algorithm for solving the job-shop problem, Management Science, **35(2)**, 164-176 **(1989)**.

[5] R.L.Conway, W.L.Maxwell, L.W.Miller; Theory of Scheduling, Addison-Wesley, Reading, Massachusetts, **(1967)**.

[6] A.S.Manne; On the job-shop scheduling problem, Operations Research, **8**, 219-223 **(1960)**.

[7] P.Nepomiastchy; Application of the penalty technique to solve a scheduling problem and comparison with combinatorial methods, Rapport de Recherche, no. 7, Institut de Recherche d'Informatique et d'Automatique, **(1973)**.

[8] R.V.Rogers; Multi-objective, Multi-stage Production Scheduling: Generalizations of the Machine Scheduling Problem, Ph.D. Thesis, University of Virginia, Charlottesville, VA, **(1987)**.

[9] H.H.Greenberg; A branch-bound solution to the general scheduling problem, Operations Research, **16(2)**, 353-361 **(1968)**.

[10] E.Balas; Discrete programming by the filter method, Operations Research, **15(5)**, 915-957 **(1967)**.

[11] D.Applegate, W.Cook; A computational study of the job-shop scheduling problem, ORSA Journal on Computing, **3(2)**, 149-156 **(1991)**.

[12] E.Balas; On the facial structure of scheduling polyhedra, Mathematical Programming Study, **24**, 179-218 **(1985)**.

[13] M.Dyer, L.A.Wolsey; Formulating the single machine sequencing problem with release dates as a mixed integer program, Discrete Applied Mathematics, **26**, 255-270 **(1990)**.

[14] A.S.Manne; On the job-shop scheduling problem, Operations Research, **8**, 219-223 **(1960)**.

[15] H.D.Sherali, B.M.P.Fraticelli; Enhancing RLT relaxations via a new class of semide_nite cuts, Journal of Global Optimization, special issue in honor of Professor Reiner Horst, P.M.Pardalos, N.V.Thoai, (Eds); **22(1-4)**, 233-261 **(2002)**.

[16] H.D.Sherali, J.Desai; On solving polynomial, factorable, and black-box optimization problems using the RLT methodology, Essays and Surveys on Global Optimization, C.Audet, P.Hansen, G.Savard, Eds., Springer, New York, NY, 131-163 **(2005)**.

[17] H.D.Sherali, S.C.Sarin, P.F.Tsai; A class of lifted path and flow-based formulations for the asymmetric traveling salesman problem with and without precedence constraints, Discrete Optimization, **3**, 20-32 **(2006)**.