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# An optimization algorithm for generalized linear multiplicative problems

**Jingben Yin*, Yongming Sun**
**School of Mathematical Science, Henan Institute of Science and Technology, Xinxiang 453003, (CHINA)**
**E-mail: jingbenyin@163.com**

## ABSTRACT

In this paper, by using new linear linearization method we present an optimization algorithm for globally solving a class of multiplicative problems which have a broad application in biotechnology, information technology, and so on. By utilizing characteristic of quadratic function, a series of linear relaxation programming problem of the initial problem can be derived and which can provide a reliable lower bound. By means of the subsequent solutions of a sequence of linear relaxation programming problems, the proposed optimization algorithm converges to the global optimal solution of the initial problem. Numerical experimental results show that the proposed algorithm is feasible and effective.

## KEYWORDS

Multiplicative problem; Optimization algorithm; New linearization method; Linear relaxation programming; Branch and bound.

## INTRODUCTION

In this paper, we shall consider the following a class of multiplicative problem:

$$\text{(MP)}:\begin{cases}\min g(x)=\sum_{j=1}^{p}(\sum_{i=1}^{n}c_{ji}x+d_j)(\sum_{i=1}^{n}e_{ji}x_i+\beta_j)\\ s.t.\ \ Ax\le b,\ \ x\in S^0=\{x\in R^n: l^0\le x\le u^0\}\end{cases}$$

where $A\in R^{m\times n}$, $b\in R^m$, $l^0=(l_1^0,l_2^0,\cdots,l_n^0)^T$, $u^0=(u_1^0,u_2^0,\cdots,u_n^0)^T$.

In last many years, many research have been done on solving the multiplicative problem (MP). On the one hand, this is since this problem (M)) exists many important applications in economic planning, engineering designing, financial plan, robust optimization [1-6], and so on. On the other hand, it is because that the problem (MP) usually poses many significant theoretical challenge and computational difficulties, i.e., it is well known that the problem (MP) possesses multiple local optimal solutions which are not globally optimal solution. For instances, when $p=1$, the problem (MP) is a special case of nonconvex programming problems, which is well known to be NP-hard [7]. Therefore, it has attracted interest of many researchers and practitioners.

Up to today, although many feasible algorithms have proposed for solving the multiplicative problem (MP), but to our knowledge, few algorithm has been still designed for globally solving the multiplicative problem (MP).

Many algorithms have proposed for linear multiplicative programming problem (MP). For example, a large number of quadratic programming methods can be obtained to solve the multiplicative problem (MP) in the literatures [8-13]. When feasible region is a polyhedral set and $p\ge 2$, the branch and bound algorithms, the approximating algorithms, the outcome space branch and bound approaches, the cutting plane methods, the heuristic methods, the monotonic optimization approaches, the simplicial branch and bound algorithms can be used to solve the problem (MP) in [14-28]. Recently, the authors in [29-32] presented several different feasible and effective algorithms for solving the generalized linear multiplicative programming problem (MP). In addition, some feasible global optimization algorithms for solving generalized nonlinear multiplicative programming problem have been proposed in [33-37].

In this article, we will present a feasible algorithm for the problem (MP) by solving a series of linear relaxation programming problems over partitioned subsets. To globally solve the problem (MP), we first transform the problem (MP) into an equivalent quadratic programming problem (EP), then, a new linearization technique is used to systematically convert the problem (EP) into a sequences of linear relaxation programming problems. The optimal solutions of these transformed problems can approximate sufficiently the global optimal solution of the problem (EP) by a successive partition process. Finally, numerical examples and their computational results are given, and numerical results show that the proposed algorithm can be used to solve all the test problems in computing the global optimal solutions of the multiplicative problem (MP) within a given tolerance condition.

The paper is described as follows. In Sections 2, first we convert the problem (MP) into an equivalent problem (EP), then a new linearizing method is proposed for generating the linear relaxation of the problem (EP). In Section 3, using the new linearizing method, a branch and bound algorithm is established for globally solving the (EP), and the convergence of the proposed algorithm is proved. Some numerical results are reported in Section 4 and Section 5 provides some concluding remarks.

## NEW LINEARIZATION METHOD

Using Firstly, we shall convert the objective function of the problem (MP) into an equivalent quadratic function.

$$\begin{aligned}g(x)&=\sum_{j=1}^{p}\left(\sum_{i=1}^{n}c_{ji}x_i+d_j\right)\left(\sum_{i=1}^{n}e_{ji}x_i+\beta_j\right)\\ &=\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\sum_{j=1}^{p}c_{ji}e_{jk}\right)x_ix_k+\sum_{i=1}^{n}\left(\sum_{j=1}^{p}d_je_{ji}+\beta_jc_{ji}\right)x_i\\ &=\sum_{j=1}^{p}\sum_{i=1}^{n}\sum_{k=1}^{n}c_{ji}e_{jk}x_ix_k\\ &\quad+\sum_{j=1}^{p}\left(d_j\sum_{i=1}^{n}e_{ji}x_i+\beta_j\sum_{i=1}^{n}c_{ji}x_i\right)+\sum_{j=1}^{p}d_j\beta_j\\ &\quad+\sum_{j=1}^{p}d_j\beta_j\end{aligned}$$

Let

$$Q_{ik}=\sum_{j=1}^{p}c_{ji}e_{jk},\ \ i=1,\ldots,n,k=1,\ldots,n,$$

$$B_i = \sum_{j=1}^{p} (d_j e_{ji} + \beta_j c_{ji}), i = 1, \ldots, n,$$

$$E = \sum_{j=1}^{p} d_j \beta_j,$$

Define the matrix $Q = (Q_{ik})_{n\times n}$, where $Q_{ik}$ is a component of $Q$, define the vector $B$, where $B_i$ is $i_{th}$ a component of $B$. Obviously, the problem (MP) can be converted into the following equivalent quadratic programming problem:

$$(\text{EP}): \begin{cases} \min g(x) = x^T Q x + B^T x + E \\ s.t.\ Ax \le b,\ \ x \in S = \{x \in R^n : l \le x \le u\}. \end{cases}$$

Let

$$S^k = \left\{x \in R^n : l^k \le x \le u^k\right\} \subseteq S^0,$$

$$l^k = \left(l_1^k, \ldots, l_n^k\right)^T,\ \ u^k = \left(u_1^k, \ldots, u_n^k\right)^T.$$

Set $\lambda_{\min}$ be the smallest eigenvalues of $Q$, and let

$$\theta = \begin{cases} 0, & if\ \ \lambda_{\min} \ge 0; \\ |\lambda_{\min}| + \tau, & if\ \ \lambda_{\min} < 0. \end{cases}$$

where $\tau \ge 0$, then $Q + \theta I$ is a positive semi-definite matrix. Then

$$\begin{aligned} g(x) &= x^T Q x + B^T x + E \\ &= x^T\left(Q + \theta I\right)x + B^T x - \theta\sum_{j=1}^{n} x_j^2 + E. \end{aligned}$$

According to the characteristic of quadratic function $-x_j^2$ over the interval $[l_j^k, u_j^k]$, we have

$$-x_j^2 \ge 2u_j^k x_j - 2u_j^k l_j^k + (l_j^k)^2,\ j = 1, \cdots, n.$$

$$\begin{aligned} x^T Q x &= (u^k - x)^T\left(Q + \theta I\right)(u^k - x)^T - \theta_i \sum_{j=1}^{n} x_j^2 \\ &\quad + 2(u^k)^T\left(Q + \theta I\right)x - (u^k)^T\left(Q + \theta I\right)u^k \\ &\ge -\theta_i \sum_{j=1}^{n} x_j^2 + 2(u^k)^T\left(Q + \theta I\right)x \\ &\quad - (u^k)^T\left(Q + \theta I\right)u^k \\ &\ge -\theta_i \sum_{j=1}^{n} [-2u_j^k x_j + 2u_j^k l_j^k - (l_j^k)^2] \\ &\quad + 2(u^k)^T\left(Q + \theta I\right)x - (u^k)^T\left(Q + \theta I\right)u^k. \end{aligned}$$

Hence, we have

$$\begin{aligned} g(x) &\ge -\theta_i \sum_{j=1}^{n} [-2u_j^k x_j + 2u_j^k l_j^k - (l_j^k)^2] \\ &\quad + 2(u^k)^T\left(Q + \theta I\right)x - (u^k)^T\left(Q + \theta I\right)u^k \\ &\quad + B^T x + E \\ &= g^L(x). \end{aligned}$$

Hence, we can establish the linear relaxation programming (RLP) of the problem (EP) over $S^k$ as follows:

$$\text{RLP}(S^k): \begin{cases} \min\ g^L(x), \\ s.t.\ Ax \le b, \\ \qquad x \in S^k = \left\{x : l^k \le x \le u^k\right\}. \end{cases}$$

Denote $v(S^k)$ and $LB(S^k)$ as the global optimal value of the problem $\text{EP}(S^k)$ and the problem $\text{RLP}(S^k)$, respectively. Obviously, we have

$$v(S^k) \ge LB(S^k).$$

Theorem 1. For any $x \in S^k$ , we have the following conclusions:

$$\lim_{\|u^k - l^k\| \to 0} \left|g(x) - g^L(x)\right| = 0.$$

Proof. By the expression of the function $g(x)$ and $g^L(x)$, for any $x \in S^k$, we can get that

$$
\begin{aligned}
g(x) - g^L(x) = {} & x^T\left(Q+\theta I\right)x + d^T x - \theta\sum_{j=1}^{n} x_j^2 \\
& -[-\theta_i\sum_{j=1}^{n}[-2u_j^k x_j + 2u_j^k l_j^k - (l_j^k)^2] \\
& + 2(u^k)^T\left(Q+\theta I\right)x - (u^k)^T\left(Q+\theta I\right)u^k \\
& + d^T x] \\
= {} & x^T\left(Q+\theta I\right)x - [2(u^k)^T\left(Q+\theta I\right)x \\
& - (u^k)^T\left(Q+\theta I\right)u^k] \\
& - \theta[(\sum_{j=1}^{n}[-2u_j^k x_j + 2u_j^k l_j^k - (l_j^k)^2) \\
& - \sum_{j=1}^{n} x_j^2] \\
\le {} & \Big\|(u^k - x)^T\left(Q+\theta I\right)(u^k - x) \\
& + \theta\sum_{j=1}^{n}[2u_j^k x_j - 2u_j^k l_j^k + (l_j^k)^2 - x_j^2]\Big\| \\
\le {} & \Big\|(u^k - x)^T\left(Q+\theta I\right)(u^k - x) \\
& + \theta\sum_{j=1}^{n}(x_j - l_j^k)(u_j^k - x_j + u_j^k - l_j^k)\Big\| \\
\le {} & \left\|Q+\theta I\right\|\left\|u^k - l^k\right\|^2 \\
& + 2\theta\left\|(u^k - l^k)^T(u^k - l^k)\right\| \\
\le {} & \left\|Q+\theta I\right\|\left\|u^k - l^k\right\|^2 + 2\theta\left\|u^k - l^k\right\|^2
\end{aligned}
$$

Therefore,

$$\lim_{\|u^k - l^k\|\to 0}\left|g(x) - g^L(x)\right| = 0.$$

Based on the above new linearization method, we can construct the linear relaxation programming (LRP) of the problem (EP), which can offer a valid lower bound for the global optimal value of the problem (EP) over rectangle $S^k$.

## BRANCH AND BOUND ALGORITHM AND ITS CONVERGENCE

Tables In this section, based on the former linearization technique, an effective branch and bound algorithm is proposed for globally solving the problem (EP). To compute the global optimization solution of the problem (EP), the proposed algorithm needs to solve a series of linear relaxation programming problem over partitioned subsets of $S^0$. Furthermore, to guarantee that the proposed algorithm is convergent to the global optimal solution.

The proposed algorithm is based on subdividing the set $S^0$ into two sub-hyper-rectangles, and each sub-hyper-rectangle is corresponding to a node of a branch and bound tree, and each node is corresponding to a linear relaxation programming problem in the associated sub-hyper-rectangle. Therefore, at iteration $k$ of the proposed algorithm, assume that we get a collection of active nodes represented as $\Omega_k$, say, each is associated with a hyper-rectangle $S \subseteq S^0,\ \forall S \in \Omega_k$. For each such node $S$, we will calculate a lower bound $LB(S)$ of the problem (EP) by solving the problem (RLP). Therefore, the lower bound of global optimal value of the problem (EP) on the whole initial rectangle $S^0$ at iteration $k$ is given by $LB_k = \min\{LB(S), \forall S \in \Omega_k\}$.

As the optimal solution of the relaxation linear programming problem (RLP) is feasible to the problem (EP), we renew the upper bound $UB_k$ of the problem (EP), if necessary. Therefore, the active nodes collection $\Omega_k$ satisfy $LB(S) < UB,\ \forall S \in \Omega_k$, at any stage $k$. We now select an active node to partition its associated hyper-rectangle into two sub-hyper-rectangles as described below, computing the lower bounds for each new node as before. Upon detecting any non-improving nodes, we can get the collection of active nodes for the next iteration, and this process is repeated until the condition of the convergence is satisfied.

The critical element in guaranteeing convergence to a global minimum is the choice of a suitable partitioning strategy. In our paper we choose a simple and standard bisection rule. This method is sufficient to ensure convergence since it drives all the intervals to zero for all variables. This branching rule is as follows.

Suppose that the rectangle

$S^k = [l^k, u^k] \subseteq S^0$

will be divided. Then we will choose the branching variable $x_p$ satisfying

$p \in \arg\max\{u^k - l^k : i = 1, 2, \cdots N\}$

and subdivide $S^k$ by partitioning the interval $[l_p, u_p]$ into the two subintervals

$[l_p, (l_p + u_p)/2]$ and $[(l_p + u_p)/2, u_p]$.

Assume that $LB(S^k)$ be the global optimal value of the problem (RLP) over the rectangle $S^k$ and suppose that $x^k = x(S^k)$ be the global optimal solution of the problem (RLP) over the rectangle $S^k$. The steps of the proposed branch and bound algorithm are given as follows.

Algorithm statement:

Step 0. (Initializing)

Let initial the iteration number $k := 0$, the initial set of the active node $\Omega_0 = \{S^0\}$; the initial upper bound $UB = \infty$, and the initial set of feasible solution $F := \varnothing$.

Compute $LB_0 := LB(S)$ and $x^0 := x(S)$ by solving the problem (RLP) over rectangle $S \subseteq S^0$.

If $x^0$ is a feasible solution of the problem (EP), we update feasible set $F$ and the upper bound $UB$, if necessary.

If $UB \le LB_0 + \varepsilon$, where $\varepsilon > 0$ is a given tolerance constant number, then terminate with $x^0$ be the optimal solution of the problem (EP). Otherwise, continue to the following Step 1.

Step 1. (Bounding) Choose the midpoint $x^{mid}$ of $S^k$, if $x^{mid}$ is a feasible solution of the problem (EP), then let $F := F \bigcup \{x^{mid}\}$.

And define the upper bound by

$UB := \min_{x \in F} g(x)$.

If $F \ne \varnothing$, denote the best known feasible solution by

$b := \arg\min_{x \in F} \varphi(x)$.

Step 2. (Subdividing) According to the proposed partitioning rule, choose the branching variable $x_p$ to subdivide the rectangle $S^k$ into two new sub-hyper-rectangles. Denote the set of new partitioned rectangles by $\bar{S}^k$.

For each rectangle $S \in \bar{S}^k$, compute the lower bound value $g^L$ of $g(x)$ over the hyper-rectangle $S$. If the lower bounds $g^L$ satisfies $g^L > UB$, then delete the corresponding sub-hyper-rectangle $S$ from $\bar{S}^k$, i.e.

$\bar{S}^k := \bar{S}^k \setminus S$,

and skip to next element of $\bar{S}^k$.

If $\bar{S}^k \ne \varnothing$, compute $LB(S)$ and $x(S)$ by solving the problem (RLP) over the rectangle $S \in \bar{S}^k$.

If $LB(S) > UB$, let $\bar{S}^k := \bar{S}^k \setminus S$; otherwise, renew the obtained $UB$, $F$ and $b$ if possible, as step 1.

Step 3. (Bounding) The remaining partition set is denoted by

$\Omega_k := (\Omega_k \setminus S^k) \bigcup \bar{S}^k$

which can give a new lower bound

$LB_k := \inf_{S \in \Omega_k} LB(S)$.

Step 4. (Termination) Detect non-improving nodes by letting

$\Omega_{k+1} = \Omega_k \setminus \{S : UB - LB(S) \le \varepsilon, S \in \Omega_k\}$.

If $\Omega_{k+1} = \varnothing$, then algorithm terminates with $UB$ be the optimal value of the problem (EP), and $b$ is the global optimal solution. Otherwise, $k := k+1$, and choose an active node $S^k$ such that

$S^k = \arg\min_{S \in \Omega_k} LB(S), x^k := x(S^k)$,

and continue to Step 1.

Theorem 2 (convergence theorem) The proposed branch and bound algorithm either stops finitely with the global optimal solution of the problem (EP), or produces an infinite iteration sequence which satisfies any limitation point of the sequence $\{x^k\}$ will be the global solution of the problem (EP) along any infinite branch of the branch-and-bound tree.

Proof. The proof of the theorem can be easily given according to Ref. [30].

## NUMERICAL EXAMPLES

To verify the effectiveness and feasibility of the proposed branch and bound algorithm, several test examples are implemented on Intel(R) Core(TM)2 Duo CPU (1.58GHZ) microcomputer, the proposed branch and bound algorithm is coded in C++ procedure, the simplex algorithm is used to solve linear relaxation programming problem and the convergence tolerance is set as $\varepsilon = 10^{-6}$. These test examples and their numerical results are given as follows.

Example 1.

$$\begin{cases} \min\ 3x_1^2 + 2x_2^2 \\ s.t.\quad x_1 + 2x_2 \le 4, \\ \qquad x_1 - 3x_2 \le 10, \\ \qquad 1 \le x_1, x_2 \le 3. \end{cases}$$

Using the proposed algorithm in this paper, the global $\varepsilon-$optimal solution $(1.0,1.0)$ and global $\varepsilon-$optimal value 5.0 is obtained.

Example 2.

$$\begin{cases} \min\ (x_1 + x_2)(x_1 - x_2) + (x_1 + x_2 + 1)(x_1 - x_2 + 1) \\ s.t.\quad x_1 + 2x_2 \le 10, \\ \qquad x_1 - 3x_2 \le 20, \\ \qquad 1 \le x_1, x_2 \le 3, \end{cases}$$

Using the proposed algorithm in this paper, the global $\varepsilon-$optimal solution $(1.0,3.0)$ and global $\varepsilon-$optimal value -13.0 is obtained.

Example 3.

$$\begin{cases} \min\ (x_1 + x_2)(x_1 - x_2) + (x_1 + x_2 + 2)(x_1 - x_2 + 2) \\ s.t.\quad x_1 + 2x_2 \le 20, \\ \qquad x_1 - 3x_2 \le 20, \\ \qquad 1 \le x_1, x_2 \le 4, \end{cases}$$

Using the proposed algorithm in this paper, the global $\varepsilon-$optimal solution $(1.0,4.0)$ and global $\varepsilon-$optimal value -22.0 is obtained.

From the numerical results for test examples 1-3, we can get that the proposed branch and bound algorithm is competitive and can be used to globally solve the generalized linear multiplicative problem (MP).

## CONCLUDIONG REMARKS

In this article, a branch and bound algorithm is proposed to solve the problem (MP). In the algorithm, a new linearization method is proposed. By utilizing the method the problem (MP) can be transformed into a series of linear relaxation programming problems, which can be used to compute the lower bound of the global optimal value of the problem (MP). The proposed branch and bound algorithm is convergent to the global optimal solution of the problem (MP) by solving a series of linear relaxation programming problems. Numerical results show the feasibility of the proposed branch and bound algorithm.

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## REFERENCES

**[1]** Konno H, Watanabe H 1994a Bond Portfolio Optimization Problems and Their Applications to Index Tracking, *J. Oper. Soc. Japan*, 39: 295-306

**[2]** Quesada I, Grossmann I E 1996 Alternative bounding approximations for the global optimization of various engineering design problems. In I.E. Grossmann, (ed.), Global Optimization in Engineering Design, *Nonconvex Optimization and Its Applications*, Kluwer Academic Publishers, Norwell, MA, 9 pp.309-331

**[3]** Maranas C D, Androulakis I P, Floudas C A, Berger A J and Mulvey J M 1997 Solving long-term financial planning problems via global optimization, *J. Economic Dynamics & Control*, 21: 1405-1425

**[4]** Henderson J M, Quandt R E 1971 *Microeconomic Theorey*. 2nd edition, McGraw-Hill, New York

**[5]** Mulvey J M, Vanderbei R J, and Zenios S A 1995 Robust optimization of large-scale systems, *Oper. Res.*, 43: 264-281

**[6]** Matsui T 1996 NP-Hardness of Linear Multiplicative Programming and Related Problems, *J.Global Optim.*, 9: 113-119

**[7]** Konno H, Kuno T 1992 Linear Multiplicative Programming, *Math. Program*., 56: 51-64
**[8]** Swarup H 1966 Programming with Indefinite Quadratic Function with Linear Constraints, *Cahier du Coutre d'Etudes de Recherche Operationelle*, 8: 223-234
**[9]** Li H M, Zhang K C 2006 A decomposition algorithm for solving large-scale quadratic programming problems, *Appl. Math. Comput.*, 173(1): 394-403
**[10]** Wu H, Zhang K 2008 A new accelerating method for global non-convex quadratic optimization with non-convex quadratic constraints, *Appl. Math. Comput*., 197(2): 810-818
**[11]** Liu S T, Wang R T 2007 A numerical solution method to interval quadratic programming, *Appl. Math. Comput.*, 189(2): 1274-1281
**[12]** Shen P, Gu M 2008a A duality-bounds algorithm for non-convex quadratic programs with additional multiplicative constraints, *Appl. Math. Comput.*, 198(1): 1-11
**[13]** Shen P, Duan Y, Ma Y 2008b A robust solution approach for nonconvex quadratic programs with additional multiplicative constraints, *Appl. Math. Comput.*, 201(1-2): 514-526
**[14]** Konno H, Fukaishi K 2000 A Branch and Bound Algorithm for Solving Low Rank Linear Multiplicative and Fractional Programming Problems, *J. Global Optim.*, 18: 283-299
**[15]** Ryoo H S, Sahinidis N V 2003 Global Optimization of Multiplicative Programs, J. Global Optim., 26: 387-418
**[16]** Xue C, Jiao H, et al. 2008 An approximate algorithm for solving generalized linear multiplicative programming, *J. Henan Normal University*, 36 (5): 13-15
**[17]** Tuy H, Nghia N D 2003 Reverse Polyblock Approximation for Generalized Multiplicative/Fractional Programming, *VN J. Math.*, 31: 391-402
**[18]** Schaible S, Sodini C 1995 Finite algorithm for generalized linear multiplicative programming, *J. Optim. Theory Appl*., 87(2): 441-455
**[19]** Gao Y, Xu C, Yang Y 2006 An outcome-space finite algorithm for solving linear multiplica- tive programming, *Appl. Math. Comput.*, 179(2): 494-505
**[20]** Oliveira R M, Ferreira P A V 2010 An Outcome Space Approach for Generalized Convex Multiplicative Programs, *J. Global Optim.*, 47: 107-118
**[21]** Ashtiani A M, Ferreira P A V 2010 Global Maximization of a Generalized Concave Multiplicative Problem in the Outcome Space, *Anais do CNMAC*, 3: 377-383
**[22]** Benson H P, Boger G M 2000 Outcome-space cutting-plane algorithm for linear multiplicative programming, *J. Optim. Theory Appl.*, 104(2): 301-322
**[23]** Liu X J, Umegaki T, and Yamamoto Y 1999 Heuristic methods for linear multiplicative programming, *J. Global Optim.*, 4(15): 433-447
**[24]** Phuong N T H, Tuy H 2003 A unified monotonic approach to generalized linear fractional programming, *J. Global Optim.*, 26: 229-259
**[25]** Chen Y, Jiao H 2009 A nonisolated optimal solution of general linear multiplicative programming problems, Comput. Oper. Res., 36: 2573-2579
**[26]** Wang C F, Li S Y, Shen P 2009 Global minimization of a generalized linear multiplicative programming, *Appl. Math. Model.*, 36(6): 2446-2451
**[27]** Kuno T, Konno H 1991 A Parametric Successive Underestimation Method for Convex Multiplicative Programming Problems, *J. Global Optim.*, 1, pp. 267-286
**[28]** Shen P, Jiao H 2006 Linearization method for a class of multiplicative programming with exponent, *Appl. Math. Comput.*, 183(1): 328-336
**[29]** Wang C F, Liu S Y 2011 A new linearization method for generalized linear multiplicative programming, *Comput. Oper.* Res., 38: 1008-1013
**[30]** Jiao H, Guo Y R, Shen P 2006 Global optimization of generalized linear fractional programming with nonlinear constraints, *Appl. Math. Comput.*, 183(2): 717-728
**[31]** Jiao H 2009 A branch and bound algorithm for globally solving a class of nonconvex programming problems, *Nonlinear Anal.-Theor.*, 70: 1113-1123
**[32]** Shen P, Bai X, Li W 2009 A new accelerating method for globally solving a class of nonconvex programming problems, *Nonlinear Anal.-Theor.*, 71(7-8): 2866-2876
**[33]** Benson H P 2008 Global Maximization of a Generalized Concave Multiplicative Function, *J. Optim. Theory Appl.*, 137: 105-120
**[34]** Konno H, Kuno T, and Yajima Y 1994b Global Minimization of a Generalized Convex Multiplicative Function, *J. Global Optim.*, 4: 47-62
**[35]** Jaumard B, Meyer C, Tuy H 1997 Generalized Convex Multiplicative Programming via Quasiconcave Minimization, *J. Global Optim.*, 10: 229-256
**[36]** Burer S, Vandenbussche D 2008 A finite branch-and-bound algorithm for non-convex quadratic programming via semidefinite relaxations. *Math. Program.*, 113: 259-282
**[37]** Chai H, Chen W 2013 Uncertainty Analysis by Monte Carlo Simulation in a Life Cycle Assessment of Water-Saving Project in Green Buildings. *Inf. Technol. J.*, 12(13): 2593-2598